THE MATHEMATICAL THEORY OF *Finance*

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REVISED EDITION

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PREFACE

In this revision there has been some simplification and other alterations in the text as well as a change in the data or other modification in many of the exercises. Lower interest rates have been employed in many instances in order to have closer agreement with those currently in use. In some cases the algebraic problems have been taken out of the general lists and put by themselves, so that they can be either omitted or given special emphasis as the teacher desires. The American Annuitants' Table of Mortality has been replaced by the 1937 Standard Annuity Mortality Table given by Frank D. Kineke in Volume 39 of the *Transactions of the Actuarial Society of America*, the table being used with Mr. Kineke's permission. Commutation Columns for 3% for this table are taken from the same source, and the same rate is used in connection with the American Experience Table.

As in the original edition the purpose has been to present a teachable text that emphasizes basic thinking and analysis of both theory and problems, and also stresses well arranged and systematic computations. As before, it is assumed that the student will not have the use of a calculating machine, but must employ logarithms. The logarithms of the investment functions, based upon Glover tables, used by permission in the first edition, are again included, and pains are taken in the examples to illustrate their use. The problems in the B lists follow closely those in the A lists and give opportunity for alternative exercises, or for additional work.

It is hoped that the book can be adapted to courses of different lengths, depending upon the material that it is desired to cover, as well as upon the mathematical preparation of the students. A knowledge of elementary algebra is assumed, though a review of exponents is given in connection with the compound interest formula. Acquaintance with quadratic equations is not necessary, and treatments of the Binomial Theorem, Arithmetic Progressions, Geometric Progressions, and Logarithms are given in the Appendix.

I wish to renew my thanks to those who helped in the original preparation of the book, as well as to those who gave suggestions for the revision, or assisted in its preparation, especially to Professor Edwin L. Godfrey, who again assisted with the proofs.

Bloomington, Indiana K.P.W.

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THE MATHEMATICAL THEORY OF *Finance*

CHAPTER I

SIMPLE INTEREST AND SIMPLE DISCOUNT

1. Interest. Interest is the payment one person makes for the use of another person's money. Thus, if Mr. Smith borrows $100 from Mr. Wilson for a year, at the end of that time he will repay the $100 plus a certain excess, which usually varies from $5 to $7. This excess is the *interest;* it is a rent that Mr. Smith pays Mr. Wilson for the use of the latter's money. Paying for the use of money is the basis of modern capitalism. The borrower may be a business organization or a government, instead of an individual. Likewise the loan may be made by a bank, a life insurance company, or some other kind of institution. Modern economy depends largely upon keeping in active operation money that people have but do not need for the time being.

When a person buys a bond from an industry, his city, or the national government, he is merely lending money to the industry, the city, or the government. The bond may not be redeemed for a number of years, perhaps ten or twenty, but on industrial bonds, and some types of government bonds, interest is paid periodically, usually every six months. On the new *savings bonds*, interest is not paid at regular intervals, but is paid in one sum at the time the bond is redeemed. The older traditional bond is the type that yields the investor a steady income on his funds, and so has advantages if the investor desires current income and is not interested only in an increase of his capital.

2. Rate of interest. It is customary to measure interest in terms of the amount earned by $100 in one year. That is, to say that the *interest rate* is *six per cent,** or 6%, means that $106 will be repaid one year after $100 has been borrowed.

---

* The full designation is *per centum*, the word *centum* being the Latin word for *hundred.*

The interest is \$6. If \$107 is paid at the end of the year, the interest rate is 7%; if \$104.50 is repaid, the rate is 4½%; etc.

The sum of money originally loaned or invested is called the *principal.*

To find the interest earned in a year, multiply the principal by the amount earned per dollar at the rate quoted.

Example 1. The interest on \$2374.92 in one year at 6% is

$$\$2374.92 \times .06 = \$142.50.$$

The rate can easily be found if the interest and the principal are given.

Example 2. Let the principal be \$4000 and the interest \$170. We have

$$\frac{170}{4000} = .0425,$$

so that the rate is 4¼%.

3. Algebraic formulas. We have so far proceeded as one does in arithmetic. We now introduce letters so as to write the algebraic formulas.

Let

i = the rate, expressed decimally, as the part of a dollar earned by a dollar in a year; e.g., $i = .06$ when the rate is 6%;

P = the principal; I = the interest.

Then

$$I = Pi. \tag{1}$$

The quantity

$$A = P + I = P + Pi = P(1 + i) \tag{2}$$

is called the *amount.*

The desirability of having a concise formula is shown in the example that follows.

Example. How much must a person lend at 4½% in order to have an amount of \$2500 at the end of the year?

Here $A = 2500$, $i = .045$, so that

$$P = \frac{A}{1 + i} = \frac{2500}{1.045} = 2392.34.$$

Thus the loan must be for \$2392.34.

4. Simple interest. Many financial transactions are for periods other than a year, and it is necessary to have a way of computing interest for such periods.

In *simple interest* it is assumed that the interest earned is proportional to the length of the period. Thus, the interest on \$100 for 6 months at 6% is \$3, since the interest is \$6 for a year.

The formulas for simple interest are as follows:

$$(3)\qquad \begin{aligned} I &= Pti, \\ A &= P + I = P(1 + ti), \end{aligned}$$

where t is the period in years. We shall use the formula only for cases where t is less than or equal to 1.

Example. Find the amount of \$250 in three months at 6% simple interest. We have $t = \frac{1}{4}$, $i = .06$, $ti = .015$,

$$A = \$250(1.015) = \$253.75.$$

EXERCISE

It is important to be able to handle algebraic formulas easily, and to change a relation from one form to another. The student is urged to work the following exercise:

From the equation

$$A = P(1 + ti)$$

express each of the three quantities P, t, and i in terms of the other three letters in the equation.*

---

* The student might like to see the formula printed in its various forms so that he can use them as occasion arises. They are omitted in order to prevent this. When one has the habit of hunting for special formulas, he has usually ceased to understand the subject thoroughly. Often he spends more time in the search than is necessary to change the basic formula. The general goal should be to know *basic* formulas, or to be able to turn to them without delay, and to have sufficient command of algebra to transform them as needed, quickly and accurately. Often the transformation should be made mentally. To make those called for in the exercise helps a little toward developing algebraic power that will be of high value throughout the work. Furthermore, the student should try to write down the two quantities t and i, as well as P, in terms of the other three quantities in the basic formula, carrying on the various steps mentally. To give the student the formulas for use in exercises would be to encourage him to become too dependent upon them. Of course, if he has the power of writing down the formulas instantly by a mere glance at the basic formula, there is no reason why he should not record them for his own use.

What has been said does not apply to a formula essentially complex, where transformations are longer. One is well justified in having such formulas in different forms. But even here the student is encouraged to have his algebra sufficiently at command to derive all forms from a basic one.

5. Ordinary and exact interest. The chief difficulty involved in simple interest is one of a purely arithmetical nature, arising from the fact that the number of days in an ordinary year, namely 365, is not a multiple of convenient periods, such as 30 days or 60 days, for which loans are frequently made. For example, it is evidently tempting to say that the interest on $100 at 6% in 60 days is $1. It would be exactly this amount if the year had 360 days instead of 365.

On account of the fact noted, there are different practices in computing simple interest.

If t is computed by using 360 days for the year, the interest is called *ordinary*, represented by I_0.

If t is computed by using 365 days for the year, the interest is called *exact*, represented by I_e.

Example 1. If the period is 60 days we have $t = 1/6$ for ordinary interest, $t = 60/365$ for exact interest.

Example 2. Find the ordinary and exact interest on $3250 for 37 days at 5%.*

$$I_0 = \$3250\left(\frac{37}{360} \times .05\right) = \$16.70,$$

$$I_e = \$3250\left(\frac{37}{365} \times .05\right) = \$16.47.$$

$$\begin{array}{rl}
\log 3250 = & 3.51\ 1883 \\
\log 37 = & 1.56\ 8202 \\
\log .05 = & 8.69\ 8970 \\
\hline
 & 3.77\ 9055 \\
\log 360 = & 2.55\ 6303 \\
\log 365 = & 2.56\ 2293 \\
\hline
\log I_0 = & 1.22\ 2752 \\
\log I_e = & 1.21\ 6762.
\end{array}$$

When the number of days is not a convenient fraction of 360, such as 30 days, 60 days, or 90 days, it is no shorter to compute ordinary interest than exact interest, especially if logarithms are used. This is clearly seen in Example 2.

Tables exist for computing ordinary interest and exact interest. Such a table is Table II at the back of the book.

Example 3. Find the ordinary interest and exact interest in Example 2 above by using the table.

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* Five-place logarithms would have been ample in this computation. Here and elsewhere the — 10 that appears as part of a negative characteristic is omitted. The student should observe that the logarithm of the product of the three numbers common to I_0 and I_e is obtained first, so that no numbers are written twice.

From the table it is seen that

	ORDINARY	EXACT
Interest on \$1000 at 1% for 30 days	.8333*	.8219
Interest on \$1000 at 1% for 7 days	.1944	.1918
Interest on \$1000 at 1% for 37 days	1.0277	1.0137

Hence

$$I_0 = \$1.0277 \times 5 \times 3.25 = \$16\ 70,$$
$$I_e = \$1.0137 \times 5 \times 3.25 = \$16.47.$$

Again, when the table is used, it is seen that ordinary interest is not simpler to calculate than exact interest.

Another point that requires care, or an agreement, is that of finding the number of days in the period. Should the exact number of days be used, or should we count by months of 30 days? For example, suppose a loan made on February 12 is repaid on October 21. How many days shall we say have elapsed? February 12 is the 43rd day and October 21 is the 294th day of the year, as is seen by consulting Table I. The exact interval in days is thus 294 − 43 = 251. (In a leap year it would be 252 days.) On the other hand, we might say that the period is 8 months and 9 days, or $30 \times 8 + 9 = 249$ days. Thus one would get two values for the ordinary interest, and two values for the exact interest, according to whether one used 251 days or 249 days for the period; or, in all, four values for the interest.

If one does not have a table such as Table I, one is likely to count by months of 30 days. The use of ordinary interest for the exact number of days is called the *Banker's Rule*.

EXERCISES

A

1. Find by direct calculation, using logarithms, the ordinary and the exact interest on \$4750 for 117 days at 6%.

2. Solve Exercise 1 by the tables

3. Find the ordinary interest on \$5360 for 68 days at 5%.

4. Find the exact interest on \$520 for 45 days at 6 3%.

---

* Only four decimals are used, on account of the small principal involved. With a little experience a person can adjust the number of figures to the requirement of the case, and thus save unnecessary arithmetic. Except when large sums are involved, results should be given correct to the nearest cent. When logarithms are used, the accuracy attainable depends on the number of places in the logarithms employed.

5. From Table I find the number of days from April 10 to December 28 of the same year.

6. Find the number of days from January 12, 1932, to May 25, 1932.

7. Find the number of days from November 17, 1930, to March 12, 1931; from August 11, 1943, to June 5, 1944.

8. A note dated January 21, 1930, was due in 90 days. On what date was it due? if dated November 18, 1935, and due in 180 days? Use the table.

9. Find the interest on a note for $800 dated January 25, 1929, and paid April 17, 1929, interest at 5½%, using the Banker's Rule.

10. Use the Banker's Rule to find the interest on $62,348 from March 15, 1945, to November 8, 1945, at 2.75%.

11. The sum of $5200 was borrowed February 5, 1937, with interest at 5%, and was paid on December 18, 1937. Calculate the exact interest.

12. A note for $5200 with interest at 6% was dated April 17, 1933. On what day should it have been paid to have the exact interest as near $100 as possible?

13. Let D be the number of days. Show that

$$I_0 = Pi\frac{D}{360}, \qquad I_e = Pi\frac{D}{365},$$

$$\frac{I_0}{I_e} = \frac{365}{360} = \frac{73}{72}, \qquad \frac{I_e}{I_0} = \frac{72}{73},$$

$$I_0 = I_e + \frac{I_e}{72}, \qquad I_e = I_0 - \frac{I_0}{73}.$$

Does the custom of using ordinary interest favor the borrower or the lender?

14. Using the formulas just established, find the ordinary interest if the exact interest is $125; the exact interest if the ordinary interest is $63.25.

15. In repaying a loan in 65 days with ordinary interest at 6% a man paid $859.21. What was the principal borrowed?

16. On January 12, 1937, a man borrowed $2000 for a year with interest at 6%, and had the privilege of partial payments. On April 15 he paid $200, on July 5 he paid $350, and on November 24 he paid $500. What did he owe when the note was due? Use exact interest.

B

1. Find by direct calculation, using logarithms, the ordinary and the exact interest on $5275 for 235 days at 5%.

2. Solve Exercise 1 by the tables.

3. Find the ordinary interest on $3249 for 63 days at 6%.

4. Find the exact interest on $700 for 45 days at 4.7%.

5. From Table I find the number of days from April 21 to November 14 of the same year.

6. Find the number of days from January 27, 1932, to July 23, 1932.

7. Find the number of days from September 5, 1929, to May 3, 1930

8. A note dated February 3, 1928, was to be paid in 90 days. On what date was it due? if dated December 9, 1943, and due in 180 days? Use the table.

9. Find the interest on a note for $950 dated March 15, 1930, and paid August 29, 1930, interest at 4¾%, using the Banker's Rule.

10. Use the Banker's Rule to find the interest on $38,642 from April 10, 1940, to December 18, 1940, at 3.25%.

11. The sum of $8750 was borrowed April 7, 1934, with interest at 6%, and was paid on December 14, 1934. Calculate the exact interest.

12. A note for $2700 with interest at 5% was dated December 10, 1930. On what date should it have been paid to have the exact interest as near $75 as possible?

13. Exercise 13, list A.

14. Using the formulas in the last exercise, find the ordinary interest if the exact interest is $85; the exact interest if the ordinary interest is $58.15.

15. In repaying a loan in 54 days with ordinary interest at 5% a man paid $931.94. What was the principal?

16. On April 20, 1932, a man borrowed $3500 for a year with interest at 5%, and had the privilege of partial payments. He paid $700 on November 18, 1932, and $1200 on January 25, 1933. What did he owe when the note was due, using exact interest?

17. If the Banker's Rule were literally followed would it be possible for a person to pay more interest for a period of less than a year than he would pay for an entire year? Discuss whether such a situation is likely to arise.

6. Simple discount. The meaning of *simple discount* and its relation to interest are frequently confused. The student should first understand typical situations that lead in a natural way to the practice of discounting.

Suppose that Mr. Smith holds a note signed by Mr. Wilson that calls for the latter to pay the former a certain sum at a future date. Suppose that prior to that date Mr. Smith needs money. It is not likely that Mr. Wilson will be willing to pay in advance, so Mr. Smith offers to sell the note to a bank. The bank will not be willing to pay Mr. Smith the full amount that will be received from Mr. Wilson upon the maturity of the note, for if that were done the bank would not be making any profit from its funds. Therefore the bank *discounts* the note. The real question is as to the basis upon which the amount of the discount is determined. The simplest and the usual procedure is to deduct a certain percentage of the maturity value of the note.

For example, suppose that the note is due in a year and is for \$1000, representing payment on a piece of property that Mr. Wilson has purchased from Mr. Smith. Let the bank deduct 6%, which is \$60, paying Mr. Smith the sum of \$940. The bank is said to discount by using a *discount rate* of 6%. We note now that the bank is making more than 6% *interest* on its funds. It earns a profit of \$60 in a year by using a principal of \$940, so that the interest rate is

$$i = \frac{60}{940} = .0638, \text{ or } 6.38\%.$$

To understand this, one needs merely to recall the definition of interest. If a person should lend \$940 at 6% interest for a year, he would not receive \$1000 at the end of that time, but by (2) only the amount

$$\$940 \times 1.06 = \$996.40.$$

Thus it is seen that a 6% *discount rate* is equivalent to a 6.38% *interest rate.*

It is of course possible to discount the note so that 6% interest would be earned. By (2) the amount to be paid for the note would be

$$P = \frac{\$1000}{1.06} = \$943.40.$$

If this were done it would be said that the note was discounted at the *interest rate* of 6%.

There are many occasions for discounting various types of commercial paper. Thus a manufacturing concern may sell a consignment of goods, the purchaser having 30 or even 60 days in which to make payment. If the manufacturer is in need of money, he sells the bill of sale to a commercial bank. The amount deducted from the face of the bill is usually determined by a discount rate.

Suppose a person goes to a bank to borrow \$100 for a year (a long period for such a loan from a bank, but chosen to illustrate the point in question) and the bank says it charges 7% interest. The bank should give the man \$100 and have him sign a note promising to pay \$107 in a year. Actually, however, the bank usually has the man sign a note for \$100 and gives him \$93, explaining that it charges interest in advance. This is a confusion of terms. The bank has really lent only \$93, for which it has charged the rate of interest

$$i = \frac{7}{93} = .0753 \text{ or } 7.53\%.$$

Or, we can say that the bank has discounted the note at a discount rate of 7%.

There should be no difficulty in distinguishing between interest and discount if it is remembered that:

$$\text{Interest rate} = \frac{\text{amount earned}}{\text{amount lent}},$$

$$\text{Discount rate} = \frac{\text{amount earned}}{\text{amount repaid}}.$$

7. Relation between discount rate and interest rate. We shall now derive general formulas that connect a discount rate and the corresponding interest rate. Let d be the discount rate, expressed as a decimal. This means that for every dollar to be received in a year the sum $1 - d$ is now paid. Thus $1 - d$ earns the sum d, so that the interest rate is

$$i = \frac{d}{1 - d}. \tag{4}$$

To find d when i is known, we solve (4) for d, the steps of the algebra being

$$i(1 - d) = d, \qquad i - id = d,$$
$$i = d + di = d(1 + i),$$

or

$$d = \frac{i}{1 + i}. \tag{5}$$

Approximate formulas are useful. By performing the division in (4), we find

$$i = d + d^2 + \frac{d^3}{1 - d}.$$

Since d is generally a small number, its third power d^3 is much smaller, and $d^3/(1 - d)$ can be neglected without much error.

Thus

(6) $$i = d + d^2, \text{ approximately.}$$

Similarly from (5),

$$d = i - i^2 + \frac{i^3}{1 + i},$$

so that

(7) $$d = i - i^2, \text{ approximately.}$$

In the formulas given, i and d may be the interest and discount rates, respectively, for *any* period, not necessarily a year.

The student should observe that formula (5) can be written down by using the meaning of i and d, quite as readily as (4).

If d is the discount rate for a year, then td is the *simple* discount rate for a fraction t of a year. Thus if the yearly discount rate is 6%, the three months discount rate is 6/4% = 1½%.

Example 1. What interest rate corresponds to a discount rate of 5%? Here $d = .05$ and (4) gives

$$i = \frac{.05}{.95} = \frac{1}{19} = .0526, \text{ or } 5.26\%.$$

The approximate formula (6) gives

$$i = .05 + .0025 = .0525, \text{ or } 5.25\%$$

Example 2. At what rate should notes be discounted if 7% interest is to be earned?

Here $i = .07$ and (5) gives

$$d = \frac{.07}{1.07} = \frac{7}{107} = .0654, \text{ or } 6.54\%.$$

The approximate formula (7) gives

$$d = .07 - .0049 = .0651, \text{ or } 6.51\%.$$

Example 3. If 60-day notes are discounted at the yearly rate of 8%, what is the equivalent 60-day interest rate?

The 60-day discount rate is 8/6% = 4/3%, taking a year as 360 days. Using $d = \frac{4}{300}$, we have from (4)

$$i = \frac{\frac{4}{300}}{1 - \frac{4}{300}} = \frac{4}{296} = .0135,$$

so that the 60-day interest rate is 1.35%.

For example, the discount on a \$300 note due in 60 days would be $300 \times \frac{4}{300} = 4$, so that \$296 would be paid for it. If \$296 is lent for 60 days at a 60-day rate of 1.35%, the amount at the end of the period will be \$296 × 1.0135 = \$300.

In Example 3 it should be noted that the required interest rate was not found by taking $\frac{1}{6}$ of the interest rate corresponding to a yearly discount rate of 8%.

EXERCISES

A

1. How much should be paid for a note with maturity value of \$2250 due in 40 days if discounted at a yearly discount rate of 5%?

In this and succeeding problems consider the year as having 360 days.

2. On January 20, 1931, A borrowed from B the sum of \$750 to be paid June 30, 1931, with yearly interest at 6%. On April 12 B sold A's note to C, who discounted it at a yearly discount rate of 7%. How much did B receive?

3. On May 12 a person paid \$400 for a note due August 31, with a maturity value of \$410. At what rate was the note discounted?

4. On what date is a note due November 25, with a maturity value of \$1060, worth \$1025 at a yearly discount rate of 7%?

5. A merchant has received a consignment of goods billed at \$925 and has 60 days in which to pay. He is offered a reduction of 4% if he pays within 10 days. At what interest rate could he afford to borrow money in order to take advantage of the 10-day reduction?

6. A merchant has a bill for goods calling for a payment of \$1250 in 90 days. He is offered a reduction of 2% if he pays within 30 days. If he can borrow money at 10% yearly interest, should he take advantage of the 30-day offer?

7. Find the interest rate that corresponds to a discount rate of 4%; the discount rate that corresponds to an interest rate of 8%. Use the exact and the approximate formulas.

8. Which is more profitable, to discount notes at 6.2%, or to lend money at 6.5% interest?

9. A man discounts 60-day paper at 2%. What is the interest rate for the period?

10. Show that the interest rate corresponding to a 3% discount rate is not half the interest rate corresponding to a discount rate of 6%. Explain this by noting that if d in (4) is multiplied by a number, the value of i is not multiplied by the same number.

11. The tax on an article is 10%, and a purchaser pays \$3.85. What would be the cost without tax?

12. By using (7) show that the tax that a person pays on an article on which the tax is 10% will be found approximately by multiplying the amount he pays by .09. Check by applying to Exercise 11.

13. If there is a 15% tax on transportation tickets, show that the tax a person has paid will be found by multiplying the cost of his ticket by $15/115 = 3/23 = .1304$, approximately.

14. A person paid \$21.57 for a ticket on which there had been a 15% tax. How much was the tax?

15. A man borrowed from a bank and actually received \$600. He signed a note due in 90 days. What was the amount he promised to pay if the bank used a yearly discount rate of 6%?

16. A man borrowed \$1000 from a bank for a year at 6%, according to the ordinary procedure: i.e., he signed a note for \$1000, the bank discounting at 6%. He had the privilege of making partial payments, and at the end of nine months he paid \$600. To what amount should the payment reduce the face of the note, if credit is given for the value of \$600 in three months at a discount rate of 6%?

17. A man borrowed money from a bank, receiving \$3000, and signed a note for \$3075 due in six months. He had the privilege of making partial payments, and at the end of two months he paid \$1000, and at the end of four months he paid \$800. How much did he owe when the note became due if the bank allowed him the same interest rate on his payments that it charged him on his loan?

B

1. How much should be paid for a note with maturity value of \$1235 due in 50 days if discounted at a yearly discount rate of 6%?

In this and succeeding problems consider the year as having 360 days.

2. On January 12, 1930, A borrowed from B the sum of \$625, to be paid May 31, 1930, with interest at 7% a year. On March 20 B sold A's note to C, who discounted it at a discount rate of 8% a year. How much did B receive?

3. On January 10 a person paid \$380 for a note due April 15 with a maturity value of \$395. At what rate was the note discounted?

4. On what date is a note due September 12, with a maturity value of \$750, worth \$735 at a yearly discount rate of 6%?

5. A merchant has received a consignment of goods billed at \$775 and has 60 days in which to pay. He is offered a reduction of 5% if he pays within 15 days. At what interest rate could he afford to borrow money to take advantage of the 15-day reduction?

6. A merchant has a bill for goods calling for a payment of \$2500 in 90 days. He is offered a reduction of 3% if he pays within 30 days. If he can borrow money at 15% a year should he take advantage of the offer?

7. Find the interest rate that corresponds to a discount rate of 5%; the discount rate that corresponds to an interest rate of 9%. Use the exact and approximate formulas.

8. Which is the more profitable, to discount notes at 7.3% or to lend money at 7.4% interest?

9. A man discounts 90-day paper at 3%. What is the interest rate for the period?

10. Show that the interest rate corresponding to a 2% discount rate is not half the interest rate corresponding to a discount rate of 4%. Explain this by noting that if d in (4) is multiplied by a number, the value of i is not multiplied by the same number.

11. The tax on a certain article is 8%, and a purchaser pays \$2 97. What would be the cost without tax?

12. By using (7) show that the tax that a person pays on an article on which the tax is 8% will be found approximately by multiplying the amount he pays by .0736. Check by applying to Exercise 11.

13. If there is a tax of 20% on transportation tickets, show that the tax a person has paid will be found by multiplying the cost of his ticket by $\frac{1}{6}$

14. A person paid \$65.52 for a ticket on which there had been a 20% tax What tax did he pay?

15. A man borrowed from a bank and actually received \$1250. He signed a note due in 60 days. What was the amount he promised to pay if the bank used a yearly discount rate of 6%?

16. A man borrowed \$1000 from a bank for a year at 7%, according to the ordinary procedure; i.e., he signed a note for \$1000 and received \$930 He had the privilege of making partial payments, and at the end of 8 months he paid \$500. To what amount should the payment reduce the face of his note, if credit is given for the value of \$500 in 4 months at a discount rate of 7% a year?

17. A man borrowed from a bank, receiving \$2500 and signing a note for \$2575 due in 6 months. He had the privilege of partial payments, and at the end of 3 months he paid \$1000 and at the end of 4 months he paid \$500. How much did he owe when the note was due if the bank allowed him the same interest rate on his payments that it charged him on his loan?

CHAPTER II

COMPOUND INTEREST

8. Definition of compound interest. It frequently happens that the interest on a loan is not paid when it is due. As an illustration, suppose that A lends B the sum of \$1000 at 6%, taking a mortgage on B's house. It is probable that A would make the loan with the expectation of using the \$60 interest received each year as income for ordinary purposes. Suppose that at the end of a year B is unable to make the interest payment that is due A. If A feels that B's difficulties are temporary, he is not likely to foreclose the mortgage, but it is certainly equitable for him to regard that he has lent B an additional \$60. Thus at the end of the next year the interest due A is \$1060 × .06 = \$63.60. If B should pay \$123.60, the indebtedness at the beginning of the third year would be the original \$1000. Should he not pay any interest, the principal for the third year would be \$1123.60, and so on.

For a number of years preceding 1933 many banks paid as much as 4% on savings accounts. As the purpose of depositors was to accumulate funds, interest was usually left, so that interest would be earned on interest. Banks, however, found it necessary either to decrease greatly the interest paid on savings accounts, or eliminate it entirely. The Defense Savings Bonds and the Savings Bonds of the United States Government have furnished a means of increasing savings through interest that in a way replaces the former profitable savings account in a bank. Interest is not paid periodically, but remains so that interest is paid on it.

When the interest that has been earned is left and added to principal, it is said that *interest is compounded.*

9. Formula for compound interest. Let P be the original principal, and i the interest rate for a year. Then by formula (2), page 2, the amount at the end of the first year is

$$A_1 = P(1 + i).$$

The principal for the second year being A_1, the amount at the end of the second year is by the above formula

$$A_2 = A_1(1 + i).$$

Using the value of A_1 we have

$$A_2 = P(1 + i)^2.$$

At the end of three years the amount is

$$A_3 = A_2(1 + i) = P(1 + i)^3.$$

Continuing in this way, we see that at the end of n years

$$A = P(1 + i)^n. \tag{1}$$

In this formula there is contained the entire subject of compound interest. The relation is in fact the foundation of the whole subject of the mathematical theory of investment, which is built almost entirely upon formula (1) with the help of a little additional and relatively simple algebraic knowledge.

Example. On January 1,* 1912, the sum of \$700 was deposited in a bank that paid 4% interest. The interest was allowed to accumulate until January 1, 1920, when the account was withdrawn. How much was withdrawn?

We have for the required amount

$$A = \$700(1.04)^8 = \$957.99,$$

the solution by logarithms being:

$$\begin{aligned} \log 1.04 &= 0.01\ 7033 \\ 8 \log 1.04 &= 0.13\ 6264 \\ \log\ 700 &= \underline{2.84\ 5098} \\ \log A &= 2.98\ 1362 \\ A &= 957.99. \end{aligned}$$

10. Compound interest tables. There is no difficulty in calculating A by means of logarithms as just shown in the example. However, there exist compound interest tables that give the value of $(1 + i)^n$ for different values of i and n, and enable the majority of problems to be solved more quickly than by direct calculation.

Table III at the end of the book is a limited table of this kind. On left-hand pages is given the value of $(1 + i)^n$; this quantity is used when the principal P is such that direct multiplication is easy. On the right-hand page is given the value of $\log (1 + i)^n$, to be used when P is such that direct multiplication would involve considerable labor.*

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* January 1 is a legal holiday but it is a convenient date, and no misunderstanding need arise from using it.

Example 1. Work the example in section 9 by the table.

From the table, $(1.04)^8 = 1.368\ 5691$.

Hence the amount withdrawn was $\$700 \times 1.36857 = \$958.00$.

Example 2. To what amount will \$24,369 accumulate in 25 years at 3½%?

$$\begin{aligned} \log (1.035)^{25} &= 0.37\ 3509 \quad \text{(from table)} \\ \log 24369 &= 4.38\ 6838 \\ \log A &= 4.76\ 0347 \\ A &= 57{,}590. \end{aligned}$$

Thus the accumulated amount is \$57,590.

Example 3. On January 1, 1910, the sum of \$1000 was deposited in a bank that paid 3½%. On January 1, 1918, the sum of \$500 was withdrawn, and the balance of the account was withdrawn on January 1, 1922. What was the amount of the final withdrawal?

Unnecessary calculation may be employed in solving this problem. The value of the account at the time of the withdrawal being $\$1000\ (1.035)^8 = \$1316.81$, there remained after the withdrawal \$816.81, which on January 1, 1922, amounted to $\$816.81(1.035)^4$. The final calculation cannot be made mentally. The entire computation, however, is simple if the final result is indicated and one of the familiar laws of exponents is used. We have for the final amount

$$\begin{aligned} &[\$1000(1.035)^8 - \$500](1.035)^4 \\ &\quad = \$1000(1.035)^{12} - \$500(1.035)^4 \\ &\quad = \$1511.07 - \$573.76 = \$937.31. \end{aligned}$$

The result in Example 3 can be regarded as the *difference* of two accounts, the *accumulated deposit*, and the *accumulated withdrawal*. When this is seen, the problem can be solved quickly.

Much computation can frequently be saved in the way illustrated in this example.

EXERCISES

A

1. Show how the accumulated amount can be taken very easily from the tables if the principal is 500, 375, 750, 1250, 2250, etc.

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* Calculating machines are tending to displace logarithms in work of the present kind. All persons, however, do not have access to a machine; therefore. logarithmic computations are illustrated in this book.

For example, to multiply a tabular value of $(1 + i)^n$ by 500, move the decimal in the tabular value three places to the right and divide by 2. The work can all be done mentally. Decimals beyond those that give cents are easily discarded. To multiply by 375, multiply by 500 as just explained, divide the result by 4, and subtract.

2. Evaluate the following, paying attention to the arithmetic hints just given:

(a) $\$200(1.05)^8$ (b) $\$375(1.07)^{12}$ (c) $\$1750(1.02)^{20}$
(d) $\$2500(1.035)^{18}$ (e) $\$75(1.025)^6$ (f) $\$12.50(1.07)^8$

Give the interpretation of each result.

3. Calculate directly by logarithms:

(a) $\$3250(1.042)^5$ (b) $\$225.54(1.0278)^4$ (c) $\$3796(1.0516)^{12}$

4. At the death of his father a son was 16 years old, and his portion of his father's estate was \$4295. He was to receive the accumulated amount when he was 21 years old. How much did he receive if the principal was accumulated at 3½%?

5. A borrowed \$1500 from B at 6%, but was unable to pay the interest for the first 4 years. How much should he then pay B in order to continue at the original principal?

6. The sum of \$25,000 was accumulated at 2% for 7 years, and then at 4% for 5 years. What was the final amount?

7. To how much would \$250 accumulate in 65 years at 3%? Recall the fact that $a^{m+n} = a^m a^n$.

8. If the population of a city increased 4% every year, what would be the population in 1920 if it was 92,400 in 1900?

9. The following deposits and withdrawals were made in a bank paying 3½%:

January 1, 1900, deposit of \$4000;
January 1, 1905, withdrawal of \$2500;
January 1, 1908, deposit of \$500;
January 1, 1910, deposit of \$200;
January 1, 1913, withdrawal of \$1000.

How much was in the account on January 1, 1920?

10. A borrowed \$2500 from B at 5%, but did not pay interest. At the end of 4 years he paid \$1200 and he cleared the debt 5 years later. What was the final payment?

B

1. Exercise 1, list A.

2. Evaluate the following, paying attention to the arithmetic hints given in Exercise 1:

(a) $\$250(1.06)^7$ (b) $\$800(1.05)^{20}$ (c) $\$2500(1.04)^9$
(d) $\$3250(1.02)^{40}$ (e) $\$75(1.07)^{10}$ (f) $\$16.67(1.05)^{21}$.

Give the interpretation of each result.

3. Calculate directly by logarithms.

(a) $\$2745(1.056)^5$ (b) $\$386.28(1.0215)^6$ (c) $\$8975(1.0475)^{12}$.

4. At the death of his father a son was 12 years old, and his portion of his father's estate was $9750. He was to receive the accumulated amount when he was 21 years old. How much did he receive if the inheritance was accumulated at 4%?

5. A borrowed $800 from B at 5%, but was unable to pay the interest for the first 3 years. How much should he then pay B in order to continue at the original principal?

6. The sum of $22,000 was accumulated at 4% for 6 years, and then at 3½% for 8 years. What was the final amount?

7. To how much would $375 accumulate in 78 years at 2½%? Recall the fact that $a^{m+n} = a^m a^n$.

8. If the population of a city increased 7% every year, what would be the population in 1925 if it was 54,600 in 1917?

9. The following deposits and withdrawals were made in a bank paying 4% interest:

January 1, 1910, deposit of $1500;
January 1, 1914, withdrawal of $800;
January 1, 1915, withdrawal of $200;
January 1, 1917, deposit of $600;
January 1, 1920, deposit of $300.

How much was in the account on January 1, 1922?

10. A borrowed $4000 from B at 6%, but did not pay interest. At the end of 5 years he paid $2500 and he cleared the debt 3 years later. What was the final payment?

11. Periods other than a year. At the beginning of algebra integral exponents are introduced, and one works with $a^2 = a\cdot a$, $a^3 = a\cdot a\cdot a$, etc. One becomes familiar with the *laws of exponents*, for example,

$$a^2a^3 = a^5, \quad (a^3)^4 = a^{12}, \quad (ab)^3 = a^3b^3.$$

Later, *nonintegral* exponents are defined; the concept at first appears difficult, but it is one of great importance. One learns that fractional exponents indicate roots, for example,

$$a^{1/2} = \sqrt{a}, \quad a^{1/3} = \sqrt[3]{a}, \quad a^{2/3} = \sqrt[3]{a^2} = \left(\sqrt[3]{a}\right)^2,$$
$$8^{1/3} = 2, \quad 32^{3/5} = \left(\sqrt[5]{32}\right)^3 = 2^3 = 8.$$

Negative exponents are also defined, for example,

$$a^{-3} = \frac{1}{a^3}, \quad 27^{-2/3} = \frac{1}{27^{2/3}} = \frac{1}{3^2} = \frac{1}{9}.$$

Summarizing, if a is any positive number, the exponential function a^n is defined for all positive and negative values of n. In addition, it is recalled that $a^0 = 1$.

The important thing is that the laws of exponents are true for *all exponents*. That is,

$$a^m a^n = a^{m+n}, \quad (a^m)^n = a^{mn}, \quad (ab)^n = a^n b^n,$$

for all values of m and n (a and b assumed positive). Thus,

$$a^{1/2} \cdot a^{1/3} = a^{1/2+1/3} = a^{5/6}, \quad (a^{1/2})^{1/3} = a^{1/6}.$$

With these facts in mind, consider the interest problem.

The value of 1 at rate i at the end of 1, 2, 3, $\cdots$ years is found by putting $n = 1, 2, 3, \cdots$ in the expression $(1 + i)^n$. Since the expression has, under the extended definition of an exponent, meaning for all values of n, we say:

The value of 1 at rate i at the end of any time t is (by the compound interest formula)

$$(1 + i)^t.$$

For instance, the value of 1 at the end of 1 month, 3 months, 6 months, 9 months, is respectively,

$$(1 + i)^{1/12}, \quad (1 + i)^{1/4}, \quad (1 + i)^{1/2}, \quad (1 + i)^{3/4}.$$

Table VIII gives values of $(1 + i)^n$ for certain fractional values of n. To indicate that the exponent is a fraction we write $(1 + i)^{1/p}$, and give the expression for $p = 2, 3, 4, 6, 12$, which provides for periods of 6, 4, 3, 2, and 1 month, respectively. The logarithms of the quantities are also given. It is seen that the table can be used for periods other than those mentioned, by using the familiar rule of logarithms. For instance, we can find $\log (1 + i)^{5/12}$ from the relation

$$\log (1 + i)^{5/12} = 5 \log (1 + i)^{1/12}.$$

The amount for a period that embraces a whole number of years and a part of another year can be found by using the rule of exponents

$$(1 + i)^{n+1/p} = (1 + i)^n (1 + i)^{1/p},$$

as illustrated on the next pages.

Example 1. Find the interest earned by \$300 in 3 months at 5% if the compound interest formula is used.

$$A = \$300(1.05)^{1/4} = \$300(1.01227) = \$303.68:$$

so that the interest is \$3.68.

Example 2. On February 1, 1921, the sum of \$500 was deposited in a bank that paid 4% interest. What was the value of the account on April 1, 1926, (a) if simple interest is assumed for the fractional period; (b) if the compound interest formula is used for the entire period?

(a) The amount on February 1, 1926, was

$$\$500(1.04)^5 = \$608.33.$$

Ordinary simple interest on \$608.33 for 2 months is \$4.06, so that the amount on the specified date was \$612.39.

(b) Since there are $5\frac{1}{6} = 31/6$ years, we have

$$A = \$500(1.04)^{31/6} = \$500(1.04)^5(1.04)^{1/6} = \$612.32,$$

the computation being:

$$\begin{array}{lll} \log 500 & = 2.69\,8970 & \\ \log (1.04)^5 & = 0.08\,5167 & \text{(from Table III)} \\ \log (1.04)^{1/6} & = 0.00\,2839 & \text{(from Table VIII)} \\ \hline \log A & = 2.78\,6976 & \\ A & = 612.32. & \end{array}$$

12. Comparison of compound and simple interest.

Interest computed by the compound interest formula for a period less than a year is not as great as interest computed by the simple interest formula. This will be proved for the case of the fraction $1/p$ of a year, where p is an integer.

The amount of 1 at the end of the period is, by the compound interest formula, $(1 + i)^{1/p}$, and by the simple interest formula, $1 + i/p$. We shall prove that

$$(1 + i)^{1/p} < 1 + \frac{i}{p}.$$

If this inequality is true, then by raising both sides to the power p we obtain

$$(1 + i) < \left(1 + \frac{i}{p}\right)^p,$$

which is also true.* Conversely, if the second inequality is

---

* It should be recalled that

$$a^p - b^p = (a - b)\,(a^{p-1} + a^{p-2}b + \cdots + ab^{p-2} + b^{p-1}).$$

The second factor on the right is positive, so that $a^p - b^p$ and $a - b$ always have the same sign. Thus if $a > b$, then $a^p > b^p$, and conversely.

true, the first one also holds. By the binomial theorem,

$$\left(1+\frac{i}{p}\right)^p = 1 + p\left(\frac{i}{p}\right) + \frac{p(p-1)}{2}\left(\frac{i}{p}\right)^2 + \cdots + \left(\frac{i}{p}\right)^p$$

$$= 1 + i + \text{quantities all positive.}$$

The truth of the second inequality is thus established, and therefore that of the first.

The proof as given shows the truth of the theorem for ½, ⅓, ¼, . . . of a year. Its truth for a special fraction of a year, not of this form, is shown in Exercise 6, page 22.

13. To find the period or the rate. There are four quantities in the basic relation (1). If any three of them are given, the fourth can be found. We have so far considered the case of finding A when P, n, and i are given. The problem of finding P when A, n, and i are given will presently be studied systematically. We now look, by means of examples, at the problem of finding n or i.

Example 1. How long will it take $2000 to amount to $3000 at 5%? We have

$$2000(1.05)^n = 3000,$$

or

$$(1.05)^n = \tfrac{3}{2} = 1.5.$$

Taking logarithms,

$$n \log (1.05) = \log 1.5.$$

But

$$\log 1.5 = .17\,6091 \qquad \log 1.05 = .02\,1189.$$

Hence

$$n = \frac{.17\,6091}{.02\,1189} = 8.310.$$

That is, 8.310 years, or 8 years and 113 days, are required.

The solution can be found approximately from the tables. We are to find n such that

$$(1.05)^n = 1.5.$$

From the tables we have

$$(1.05)^8 = 1.4775, \qquad (1.05)^9 = 1.5513.$$

Interpolating,

$$\frac{n-8}{9-8} = \frac{1.5 - 1.4775}{1.5513 - 1.4775};\ n = 8 + \frac{.0225}{.0738} = 8.305.$$

The time required is thus 8 years and 111 days. An error of 2 days is present in the tabular solution.

The problem of finding the number of days in excess of 8 years if simple interest is used for the fraction of a year is considered in Exercise 7, page 22.

Example 2. A man bought a lot for \$2000 and sold it in 8 years for \$8000. What interest had he made on his investment?

We have

$$2000(1+i)^8 = 8000,$$

or

$$(1+i)^8 = 4.$$

Hence

$$8 \cdot \log (1+i) = \log 4 = .60\ 2060,$$

so that

$$\log (1+i) = .07\ 5258.$$

Finding the corresponding number,

$$1 + i = 1.189.$$

Thus interest at the rate of 18.9% had been realized on the investment. (It would be necessary to take taxes and assessments for improvements into account in order to find the net profit.)

EXERCISES

A

1. Obtain the values of the following:

(a) $\$800(1.06)^{1/3}$ (b) $\$12,000(1.03)^{1/2}$ (c) $\$300(1.05)^{1/4}$

(d) $\$273.64(1.02)^{1/2}$ (e) $\$7250(1.04)^{3/4}$ (f) $\$37,900(1.06)^{5/6}$.

Give the interpretation of each result.

2. Find the interest earned by \$2500 in 6 months at 6% if the compound interest formula is used; by \$2640.50 in 9 months at 4.38%.

3. On February 10, 1920, A lent B the sum of \$2500 at 6%. No interest was paid by B, but on May 10, 1923, B paid the entire obligation. Find the amount paid (a) if ordinary simple interest was used for the fractional period; (b) if the compound interest formula was used for the entire period. Count fractions of a year in months.

4. A man borrowed \$1000 for a year at 6% with the privilege of making partial payments. At the end of 9 months he paid \$700. Find the final payment if interest for a part of a year is calculated by the compound interest formula.

Consult Example 3, page 16, and the statement at the top of page 19.

5. A man borrowed \$2500 for a year at 7% with the privilege of making partial payments. He paid \$800 at the end of 6 months, and \$500 at the end of 8 months. How much did he owe at the end of the year?

6. Show that interest for two-thirds of a year is less by the compound than by the simple interest formula by proving that

$$(1+i)^{2/3} < 1 + \tfrac{2}{3}i.$$

SUGGESTION: Raise both sides to the third power and use the note on page 20.

7. In Example 1, section 13, find the number of days in excess of 8 years if exact simple interest is used for the fraction of a year.

SUGGESTION: Show that the amount at the end of 8 years is \$2954.91, and thus if D is the number of days we have the equation

$$2954.91 \frac{D}{365} (.05) = 45.09.$$

Is the computation simpler than that when compound interest is used?

8. How long does it take a sum of money to double itself at 5%? at 7%? Solve both by direct calculation and by the tables.

9. How long is required for \$2500 to amount to \$4000 at 4½%? Use both simple and compound interest for the fraction of a period.

10. A debt of \$325 contracted October 25. 1925, with accumulated interest at 6% was discharged by a single payment of \$420. When was the settlement made?

11. A man paid \$475 for a lot, and sold it in 6 years for \$700. What interest had he made on his money?

12. At what rate will \$1250 amount to \$1540 in 5 years? Solve by calculation and also approximately by interpolation in the table.

13. A United States Savings Bond, Series E, bought for \$75 is redeemable in 10 years at \$100. What is the yearly interest rate? If the rate is assumed constant, show that the value of the bond at the end of 7 years is \$75 $(4/3)^{7/10}$, and evaluate. Find the value of the bond at the end of 5 years. (For comments, see page 92.)

14. It is interesting to know the period in which a sum of money will double itself at compound interest.

The basic relation is

$$(1 + i)^n = 2,$$

so that

$$n = \frac{\log 2}{\log (1 + i)}.$$

Now log 2 = .3010, and in the calculus it is shown that

$$\log (1 + i) = .4343 \left(i - \frac{i^2}{2} + \frac{i^3}{3} - \cdots \right),$$

provided i is between -1 and 1. In interest problems i is positive and decidedly smaller than 1. Thus we have with considerable accuracy

$$\log (1 + i) = .4343 \left(i - \frac{i^2}{2}\right) = .4343i \left(1 - \frac{i}{2}\right).$$

Hence show that approximately*

$$n = \frac{.3010}{.4343i \left(1 - \frac{i}{2}\right)} = \frac{.693}{i} \left(1 + \frac{i}{2}\right) = \frac{.693}{i} + .35.$$

It should be appreciated that the result obtained is really quite simple.

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* Consult page 9 to justify taking

$$\frac{1}{1 - \frac{i}{2}} = 1 + \frac{i}{2}, \text{ approximately.}$$

15. Find the time required for a sum of money to double itself at 3%, 5%, and 7%, using the formula in Exercise 14 and also by direct calculation.

16. Show that the time in years required for a sum of money to triple itself at compound interest is given by

$$n = \frac{\log 3}{\log (1 + i)} = \frac{1.099}{i} + .55, \text{ approximately.}$$

B

1. Obtain the values of the following:

(a) $\$500(1.05)^{1/4}$	(b) $\$250(1.06)^{1/2}$	(c) $\$7500(1.03)^{1/4}$
(d) $\$250(1.06)^{1/2}$	(e) $\$3750(1.06)^{2/3}$	(f) $\$1350(1.06)^{11/12}$.

Give the interpretation of each result.

2. Find the interest earned by $2500 in 3 months at 7% if the compound interest formula is used; by $3750.50 in 8 months at 5.27%.

3. On May 20, 1924, A loaned B the sum of $2500 at 6%. No interest was paid by B, but on August 30, 1928, B paid the entire obligation. Find the amount paid (a) if simple interest was used for the fractional period; (b) if the compound interest formula was used for the entire period. Count fractions of a year in months.

4. A man borrowed $1200 for a year at 7%, with the privilege of making partial payments. At the end of 8 months he paid $800. Find the final payment if interest for a part of a year is calculated by the compound interest formula.

Consult Example 3, page 16, and the statement that follows it.

5. A man borrowed $3000 for a year at 6%, with the privilege of making partial payments. He paid $1200 at the end of 6 months and $800 at the end of 9 months. How much did he owe at the end of the year? Use compound interest.

6. Show that interest for three-fourths of a year is less by the compound than by the simple interest formula by proving that

$$(1 + i)^{3/4} < 1 + \frac{3}{4} i.$$

SUGGESTION: Raise both sides to the fourth power and use the note on page 20.

7. Exercise 7, list A.

8. How long does it take a sum of money to double itself at 4%? at 8%? Solve both by direct calculation and by the tables.

9. How long is required for $3000 to amount to $4500 at 4%? Use both simple and compound interest for the fraction of a period.

10. A debt of $750 contracted April 10, 1910, with accumulated interest at 6% was discharged by a single payment of $840. When was the settlement made?

11. A man paid $500 for a lot, and sold it in 8 years for $725. What interest had he made on his money?

12. At what rate will \$1500 amount to \$1800 in 4 years? Solve by calculation and also approximately by interpolation in the table.

13. A United States Savings Bond, Series F, bought for \$74 is redeemable in 12 years at \$100. What is the yearly interest rate if it is assumed to be constant? Show that the value of the bond at the end of 5 years would be $\$74\ (100/74)^{5/12}$ if the rate were constant. Evaluate.

14, 15, 16. See list A.

14. Nominal rate of interest. Although in the derivation of (1) we assumed that i was the yearly rate, it is seen that the significant thing is that i is the *rate per conversion period*, and that n is the *number of conversion periods*. Sometimes interest is compounded more often than once a year. A savings bank for instance may quote a rate of 3% compounded semiannually. This is described as a *nominal* rate of 3% compounded semiannually. Similarly, a nominal rate of 4% compounded quarterly means that 1% is allowed every three months.

The amount in the case of a nominal rate is readily found by (1).

Example. Find the amount of \$1250 for 12½ years at 4% nominal, compounded quarterly.

There being 50 conversion periods with a rate of 1% each period, we have

$$A = \$1250(1.01)^{50} = \$2055\ 79,$$

the computation being:

$$\begin{aligned} 1000(1.01)^{50} &= 1644.63 \quad \text{(from table)} \\ 250(1.01)^{50} &= \underline{\ 411.16\ } \quad (\tfrac{1}{4} \text{ of } 1644\ 63). \\ &\ 2055.79 \end{aligned}$$

15. Relation between nominal and effective rates. The year being a standard unit of time, it is natural to determine the interest actually earned in a year when a nominal rate compounded more than once a year is quoted. For example, with a nominal rate of 6% compounded quarterly, \$100 amounts at the end of a year to $\$100(1.015)^4 = \$106.136$, as is seen by consulting the table. The interest earned, namely \$6.136, is the same as that earned by \$100 at 6.136% computed yearly. We say that the *effective rate* is 6.136%. Similarly the effective rate corresponding to 5% nominal, compounded twice a year, is

$$(1.025)^2 - 1 = .0506, \text{ or } 5.06\%.$$

In general, let $j_{(p)}$ denote a nominal rate compounded p times a year. Let i be the equivalent effective rate, then

$$1 + i = \left(1 + \frac{j_{(p)}}{p}\right)^p, \tag{2}$$

for each side of the equation represents the value of 1 at the end of a year. From this equation i can be found in terms of j, and j can be found in terms of i, both of which relations are important. For the effective rate in terms of the nominal we have

$$i = \left(1 + \frac{j_p}{p}\right)^p - 1, \tag{3}$$

while for the nominal rate in terms of the effective, we have,

$$j_p = p[(1 + i)^{1/p} - 1], \tag{4}$$

as the student should show.

Example. What nominal rate compounded quarterly is equivalent to an effective rate of 5%?

We have

$$j_{(4)} = 4\{(1.05)^{1/4} - 1\}.$$

The result can be found by taking the value of $(1.05)^{1/4}$ from Table VIII. We have

$$j_{(4)} = 4(1.01227 - 1) = .04908;$$

that is, the nominal rate is 4.908%.

Values for nominal rates corresponding to given effective rates are given in Table IX.

16. Instantaneous conversion. The question arises as to what takes place if with a fixed nominal rate the number of conversion periods in a year is increased indefinitely. As we increase the number of conversion periods, the corresponding effective rate increases. Will it increase without limit, or will it approach a definite limit? If it approaches a limit, what will that limit be?

To answer the questions raised, it is sufficient to know what happens to the quantity

$$\left(1 + \frac{1}{n}\right)^n$$

when the number n increases without limit. It can be shown that this quantity approaches a limit.* Simple calculations lead one to surmise the truth of the statement. Putting $n = 1, 2, 3, 4 \cdots$, we have successively,

$$2,\quad \left(\frac{3}{2}\right)^2 = \frac{9}{4},\quad \left(\frac{4}{3}\right)^3 = \frac{64}{27},\quad \left(\frac{5}{4}\right)^4 = \frac{625}{256},\quad \cdots.$$

The numbers are increasing, but not rapidly so, and approach to a limit is suggested.

The limit that is approached by the quantity as n increases without limit is denoted by e, a number almost as important—if not equally so—in mathematics as the number $\pi = 3.14159 \cdots$, which is introduced as a limit in geometry. Thus we have

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e.$$

Numerically,

$$e = 2.71828\cdots, \quad \text{and} \quad \log e = 0.43\,4294.$$

Now consider a nominal rate j compounded p times a year. The amount at the end of a year is

$$\left(1 + \frac{j}{p}\right)^p.$$

Let the number n be defined by the relation

$$p = jn.$$

The amount in question is then

$$\left(1 + \frac{j}{jn}\right)^{nj} = \left[\left(1 + \frac{1}{n}\right)^n\right]^j.$$

If p increases indefinitely the number n does so also, and the limit of the last quantity is e^j.

The effective rate that corresponds to continuous conversion is thus

$$i = e^j - 1. \tag{5}$$

If interest were compounded a very large number of times, say every hour, or every minute, the resulting effective rate

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[*] An accurate proof is not elementary, and can be found in books on advanced calculus.

would be very close to that given by (5). No frequency of conversion will produce an effective rate in excess of that amount.

If i is given, the nominal rate j, which by continuous conversion would be equivalent to i, is called the *force of interest.* It can be obtained from (5).

Example 1. Find the effective rate that is equivalent to a nominal rate of 5% converted instantaneously.

We must calculate $e^{.05}$.

$$\log e = .43\,4294$$
$$.05 \log e = .02\,1715$$
$$e^{.05} = 1.05127, \qquad i = .05127.$$

This shows that no matter how frequent is the conversion at 5% nominal, the effective rate will not exceed 5.13%.

Example 2. What is the force of interest corresponding to 6% effective?

We have to solve for j the equation

We have $$e^j = 1\,06.$$
$$j \cdot \log e = \log 1.06,$$
$$j = \frac{.02\,5306}{.43\,4294} = .0583.$$

Thus the force of interest is 5.83%. That is, if one took this as a nominal rate, and compounded very frequently, for instance 1000 times in a year, he would be realizing essentially 6%.

17. Use of the binomial theorem. Interest tables and logarithms are normal equipment for dealing with compound interest problems. The student will see at once that a great deal of time and labor would be involved in calculating the value of such a quantity as $(1.04)^8$ by ordinary arithmetic, and much of the work would be wasted through calculating figures that are later thrown away as unnecessary. If one must work a compound interest problem without interest tables or logarithms, the binomial theorem should be used. (See Appendix I.)

We have

$$(6) \qquad (1+i)^n = 1 + ni + \frac{n(n-1)}{1 \cdot 2} i^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} i^3 + \cdots,$$

or, since n is an integer,

$$(7) \qquad (1+i)^n = 1 + ni + {}_nC_2 i^2 + {}_nC_3 i^3 + \cdots,$$

where ${}_nC_r$ denotes the number of combinations of n things r at a time, given by

$$(8) \qquad {}_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots r}.$$

The interest rate being a small number, the powers $i^2, i^3, \cdots$ decrease rapidly. Unless n is a large number, a relatively few terms of the expansion will give the value of $(1 + i)^n$ correct to three or four decimal places. Such a result will be sufficient unless a large principal P is involved. In the latter case interest tables or logarithms are quite indispensable.

Example. Find $(1.04)^8$ correct to three decimal places.

We arrange the computation in a column for convenience in adding the different terms:

$$\begin{array}{lllr}
(1.04)^8 = 1 & & & 1 \\
\quad + 8 \times .04 & & & .32 \\
\quad + {}_8C_2(.04)^2 & 28 \times .0016 & = & .0448 \\
\quad + {}_8C_3(.04)^3 & 56 \times .000\,064 & = & .0036 \\
\quad + {}_8C_4(.04)^4 & 70 \times .000\,00256 & = & .0002 \\
\hline
 & & & 1.369
\end{array}$$

The result 1.369 is correct to three decimals as is seen by referring to the tabular value.

18. Comment on interest formulas.

If t is time in terms of a year, and i is the interest on 1 for a year, the amount of 1 at the end of time t is, by the simple interest formula,

$$(9) \qquad 1 + ti,$$

whereas by the compound interest formula it is

$$(10) \qquad (1 + i)^t.$$

The expression (9) is a *first degree* (or *linear*) function of t, the simplest function in mathematics, while (10) is an *exponential* function, a much more complex idea, that presupposes more mathematical maturity.

But in spite of the apparent simplicity of (9) the function (10) has a great merit compared with (9) in the interest problem, which arises from the fact that the product of (9) for two different values of t is not an expression of that form; the product of (10) for two different values of t is an expression of form (10) (possessing another important property). This will be proved.

Let t_1 and t_2 be two values of t; the product of $1 + t_1 i$ and $1 + t_2 i$ is

$$(1 + t_1 i)(1 + t_2 i) = 1 + (t_1 + t_2)i + t_1 t_2 i^2,$$

and the quantity obtained is not of form (9). On the other hand

$$(1 + i)^{t_1}(1 + i)^{t_2} = (1 + i)^{t_1 + t_2},$$

and the quantity obtained is of the form (10); and the exponent is furthermore the sum of the exponents in the two quantities multiplied.

To show the advantage that the property mentioned has in the interest problem, suppose that a partial payment A' has been made at the end of time t_1 on an original loan A. The amount owed at the end of time t_2 after the payment is, by the simple interest formula,

$$\begin{aligned}(11)\quad & [A(1 + t_1 i) - A'](1 + t_2 i) \\ & = A[1 + (t_1 + t_2)i] - A'(1 + t_2 i) + A t_1 t_2 i^2,\end{aligned}$$

whereas by the compound interest formula it is

$$\begin{aligned}(12)\quad & [A(1 + i)^{t_1} - A'](1 + i)^{t_2} \\ & = A(1 + i)^{t_1 + t_2} - A'(1 + i)^{t_2}.\end{aligned}$$

The full appreciation of what has been obtained requires a little thought; it comes from a consideration of the term $A t_1 t_2 i^2$, which destroys simplicity in the first result. Consult the remark on page 16.

By the compound interest formula the interest earned in a day steadily increases, and is proportional to the accumulated amount at the moment; by the simple interest formula the daily earning is constant. The student should prove this.

EXERCISES

A

1. Find the amount of \$1125 for 10 years at 3% nominal compounded quarterly.

2. A holds B's note for \$2000 due in 3 years. He offers it to C for \$1750. Does the offer present C any advantage over depositing in a bank that pays 4% nominal compounded semiannually?

3. A deposited \$1000 in a bank that paid 3% nominal compounded annually. B deposited money in a bank that paid 2% nominal compounded quarterly. At the end of 9 years B had as much as A had at the end of 10 years. How much did B deposit?

4. In what length of time will \$300 amount to \$500 at 4% compounded semiannually?

5. What would be the effective rate if \$100 accumulated to \$150 in 10 years when compounded twice a year? A bank compounds quarterly, but at a nominal rate that is equivalent to 4% effective. What is the value of a deposit of \$100 at the end of 6 months?

6. What is the effective 6-months rate of a 4% nominal yearly rate compounded quarterly?

7. If interest is compounded monthly, what nominal rate is equivalent to an effective rate of 5%? of 2.25%?

8. What effective rate corresponds to a monthly rate of .5%? of .25%?

9. How long will it take \$1200 to amount to \$2000 at 3% nominal compounded semiannually?

10. How long is required for \$1360 to amount to \$1800 at 4% nominal compounded quarterly?

11. At what nominal rate would \$1000 compounded monthly amount to \$1200 in 5 years? What is the effective rate?

12. Could \$1200 earn \$73 in a year at 6% nominal if converted frequently enough? \$74? \$75?

SUGGESTION: Find the interest earned at the effective rate corresponding to continuous conversion.

13. Could \$1000 earn \$70 in a year if interest is converted frequently enough at 6.9%? at 6.8%? at 6.7%?

SUGGESTION: Use the force of interest corresponding to 7% effective.

14. Use the binomial theorem to find the amount of \$100 in 5 years at 4%.

15. Find $(1.02)^{15}$ correct to three decimals by the binomial theorem.

16. The sum of \$1000 was borrowed at 6%, and \$400 was paid on the account at the end of 4 months; find the amount owed at the end of 9 months using simple interest and (11).

B

1. Find the amount of \$850 for 12 years at 2½% nominal converted quarterly.

2. A holds B's note for \$3500 due in 4 years. He offers to sell the note to C for \$2900. Does the offer present C any advantage over depositing in a bank that pays 4% nominal compounded quarterly?

3. A deposited \$700 in a bank that paid 3% nominal compounded semiannually. B deposited some money in a bank that paid 2% nominal compounded quarterly. At the end of 8 years B had as much as A had at the end of 6 years. How much did B deposit?

4. In what length of time will \$250 amount to \$400 at 4% nominal compounded quarterly?

5. What would be the effective rate if \$200 accumulated to \$250 in 8 years when compounded twice a year? A bank compounds quarterly, but at a nominal rate that is equivalent to 5% effective. What is the value of a deposit of \$250 at the end of 9 months?

6. What is the effective 6-months rate of a 5% nominal yearly rate compounded quarterly?

7. If interest is compounded monthly, what nominal rate corresponds to an effective rate of 4%? of 2.75%?

8. What effective rate corresponds to a quarterly rate of 1.5%?

9. How long will it take \$2000 to amount to \$3200 at 6% nominal compounded semiannually?

10. How long is required for \$725 to amount to \$1200 at 5% nominal compounded quarterly?

11. At what nominal rate would \$800 compounded monthly amount to \$1025 in 6 years? What is the effective rate?

12. Could \$1500 earn \$76 in a year at 5% nominal, if converted frequently enough? \$76.75? \$77?

SUGGESTION: Find the interest earned at the effective rate corresponding to continuous conversion.

13. Could \$1000 earn \$60 in a year if interest is converted frequently enough at 5.9% nominal? at 5.8%?

SUGGESTION: Use the force of interest corresponding to 6% effective.

14. Use the binomial theorem to find the amount of \$200 in 8 years at 3%.

15. Find $(1.025)^{12}$ correct to three decimals by the binomial theorem.

16. The sum of \$2000 was borrowed at 5%, and \$700 was paid on the account at the end of 3 months; find the amount owed at the end of 8 months using simple interest and (11).

19. Present value. Attention has already been called to the fact that the basic equation (1), namely,

$$A = P(1 + i)^n, \tag{13}$$

contains four quantities, and that if any three of them are given, the remaining one can be obtained. We have so far considered in turn A, n, and i as the unknowns. Direct calculation is possible by the use of logarithms. Except for unusual rates or periods, A can, however, be taken from tables, and as has been seen, such tables will frequently give also an approximate solution for n or i when either of these quantities is the unknown.

We come to the remaining case, that in which P is the unknown. Table III can also be used in this instance, as an example will show.

Example 1. What principal must be invested at 5% to accumulate to \$1000 in 10 years?

Here we have

$$P(1.05)^{10} = 1000,$$

or

$$P = \frac{1000}{(1.05)^{10}} = \frac{1000}{1.628\,8946}.$$

We would now have a tedious calculation to perform. It can be avoided by using logarithms, taking log $(1.05)^{10}$ from Table III, instead of $(1.05)^{10}$.

$$\begin{aligned}
\log 1000 &= 3.00\,0000 \\
\log (1.05)^{10} &= 0.21\,1893 \\
&= \overline{} \\
\log P &= 2.78\,8107 \\
P &= \$613.91.
\end{aligned}$$

A special table, however, is convenient for the problem of finding the original principal P.

We can write formula (13) in the form

$$P = A\,\frac{1}{(1+i)^n} = A(1+i)^{-n}. \tag{14}$$

Let us introduce for convenience the letter v, where by definition

$$v = \frac{1}{1+i}. \tag{15}$$

We can then write

$$P = Av^n. \tag{16}$$

Table IV gives powers of v for different values of n and i. The numbers given in it are merely the reciprocals of those on the left-hand pages in Table III. With these numbers the value of P is given by a multiplication, as shown by (16). In the cases where A is a round number, as it frequently is, the product can be written down quickly. If this is not the case, that is, if A contains a number of figures not ciphers, the multiplication in (11) may be as tedious as the division in (9). The use of logarithms is then desirable. It should be seen that there is no need to have a table for log v^n, for it is just as easy to subtract log $(1 + i)^n$ from log A, as to add log v^n to log A.

Example 2. What sum of money would amount to \$2000 in 6 years at $4\frac{1}{2}\%$?

We have

$$P = \$2000\ v^6\ (4\tfrac{1}{2}\%).$$

From Table IV,

$$v^6(4\tfrac{1}{2}\%) = .767\ 8957.$$

Hence

$$P = \$1535.79.$$

Example 3. On his 16th birthday a son's inheritance of his father's estate was put in a trust fund that earned 4%. On his 21st birthday the son received \$3489.12. What was the original inheritance?

Here we have

$$A = \$3489.12, \qquad P = \frac{A}{(1.04)^5},$$

$$\begin{aligned} \log A &= 3.54\ 2726 \\ \log (1.04)^5 &= 0.08\ 5167 \quad \text{(from Table III)} \\ \hline \log P &= 3\ 45\ 7559 \\ P &= \$2867.87. \end{aligned}$$

The quantity $v^n = 1/(1+i)^n$ is said to be the *present value* of 1 due in n years at rate i. Problems such as those considered above are generally described as present value problems. For example, the present value at a $4\frac{1}{2}\%$ basis of \$2000 due in 6 years is \$1535.79, for this sum at compound interest at $4\frac{1}{2}\%$ will amount to \$2000 in 6 years, as shown in the example above. In the second example the same formula that is used in finding the present value of a sum due in the future was employed.

When fractional periods are involved, Table VIII can be employed.

Example 4. Find the value on January 1, 1920, of \$3000 due April 1, 1925, on a 6% basis.

The interval involving $5\frac{1}{4}$ years, we have

$$P = \frac{\$3000}{(1.06)^{5\frac{1}{4}}} = \frac{\$3000}{(1.06)^5(1.06)^{1/4}} = \$2209.36,$$

the calculation by logarithms being:

$$\begin{aligned} \log (1.06)^5 &= 0.12\ 6529 \quad \text{(Table III)} \\ \log (1.06)^{1/4} &= 0.00\ 6326 \quad \text{(Table VIII)} \\ \hline &\ 0.13\ 2855 \\ \log 3000 &= 3.47\ 7121 \\ \hline \log P &= 3\ 34\ 4266 \end{aligned}$$

The computation is shorter if made with cologarithms.

$$\begin{array}{ll} \log 3000 & = 3.47\,7121 \\ \operatorname{colog}(1.06)^5 & = 9.87\,3471 \\ \operatorname{colog}(1.06)^{1/4} & = 9.99\,3674 \\ \hline \log P & = 3.34\,4266 \end{array}$$

20. Comment. To find the present value of a sum of money due at some future date is, of course, merely to discount it, using an interest rate and not a discount rate. The quantities to be first determined are the amount that is actually to be paid in the future, the interest rate to be used in making the calculation, and the interval that is to elapse before the payment falls due.

Often the payment that is to be finally received is definitely stated. For example, A may hold B's note for \$2000, due in 4 years, the note representing a payment on a piece of property. Such a note is a *non-interest-bearing note*, and the sum to be paid when the note is due is the face value of the note. On the other hand B might have borrowed \$2000 and given a note to A that calls for the payment in 4 years of the principal and accumulated interest at 5%. The maturity value of this note would be $\$2000(1.05)^4 = \$2431.01$. If a third person C bought the note from A, it is the latter sum that he would discount.

The rate to be used in finding the present value is often specified by saying that *money is worth* a certain per cent. Such a statement implies that at a given time a certain rate of interest is generally asked or can be obtained. Within certain limits this is true. Interest rates fluctuate, and the type of security and the length of the period are also to be considered. At normal times there will be certain amounts of money seeking short period investment, other amounts seeking investment for a long period. And there will also be persons who have notes due in the future which they are either forced or, for one reason or another, merely desire to sell. The relative abundance of money to be invested and of notes, and so on, for sale will influence interest rates, so that the phrase *money being worth* has come into use.

It is to be observed that the words *present value* are frequently used in a sense not literally exact, since the date for which the value of some transaction is calculated may be prior or subsequent to the actual present moment. But it is always prior to the date when the obligation considered falls due.

EXERCISES

A

1. Find the present value of.

(a) \$3000 due in 4 years without interest, on a 2% basis.
(b) \$1500 due in 5 years without interest, on a 5% basis.
(c) \$450 due in 8 years without interest, on a 4% basis.

2. On January 1, 1918, a man borrowed \$2500, giving a note to pay the principal and accumulated interest at 6% on January 1, 1924. On January 1, 1920, he sold the note, the purchaser giving an amount such that he would earn 5% on his money. What was the selling price of the note?

Solution: Selling price

$$P = \frac{\$2500(1.06)^6}{(1.05)^4} = \$2917.55.$$

The computation should be made by logarithms (or a machine).

3. Find the value:

(a) On April 1, 1925, of \$195 due April 1, 1927, bearing interest at 6% since April 1, 1922, money being worth 5%.

(b) On July 1, 1930, of \$1250 due July 1, 1933, bearing interest at 4% since April 1, 1927, money being worth 5½%.

(c) On May 1, 1917, of \$700 due July 1, 1919, bearing interest at 5% since January 1, 1915, money being worth 4%.

(d) On January 1, 1910, of \$6000 due January 1, 1914, bearing interest at 6% since January 1, 1908, money being worth 6%.

4. Discount \$275 for 5¾ years at 2% compounded quarterly. What is the amount of the discount?

5. How much must be deposited on January 1, 1922, in a bank that pays 4% nominal compounded semiannually, in order to draw out \$2500 on January 1, 1928? How much to draw out \$3000 on July 1, 1932?

6. A person is offered a piece of property for \$3500 cash and \$1000 at the end of each year for 4 years. What is the equivalent cash price on a 5% basis?

7. A person is offered a piece of property for \$10,600 cash, or for \$7000 cash and payments of \$2000 at the end of 1 year and 2 years. If he can afford to pay cash, but can invest money at 4% nominal converted semiannually, should he pay cash?

8. A gave B a note for \$286.50 with interest at 5% nominal converted quarterly, the note being due in 2 years. At the end of 6 months C bought the note on a 6% effective basis. How much did C pay? Did B gain or lose by selling the note?

9. At the birth of his son a father wished to deposit in a bank paying 3% compounded semiannually a sum such that he could draw out $800 on the son's 17th, 18th, 19th, and 20th birthdays, in order to provide for his education. How much was it necessary to deposit?

10. Of 89,032 persons living at age 25 there will be, according to the American Experience Table of Mortality, 69,804 living at age 50 What equal payment by each member of the original group will provide for a payment of $1000 to each of the survivors at age 50, if the funds are invested at 3%?

B

1. Find the present value of:

(a) $4000 due in 3 years without interest, on a 6% basis.
(b) $1250 due in 5 years without interest, on a 4% basis.
(c) $750 due in 6 years without interest, on a 1½% basis

2. On January 1, 1910, a man borrowed $4200, giving a note to pay the principal and accumulated interest at 5% on January 1, 1918. On January 1, 1915 he sold the note, the purchaser giving an amount such that he would earn 6% on his money. What was the selling price of the note?

SOLUTION: Selling price

$$P = \frac{\$4200(1.05)^8}{(1.06)^3} = \$5210.10.$$

The computation should be made by logarithms (or a machine).

3. Find the value:

(a) On May 1, 1920, of $210 due May 1, 1923, bearing interest at 6% since May 1, 1917, money being worth 5%.

(b) On August 1, 1928, of $850 due August 1, 1930, bearing interest at 4% since May 1, 1925, money being worth 6%.

(c) On March 15, 1920, of $12,000 due March 15, 1922, bearing interest at 4½% since December 15, 1916, money being worth 5%.

(d) On December 20, 1918, of $5000 due December 20, 1924, bearing interest at 5½% since June 20, 1913, money being worth 5%.

4. Discount $325 for 7¼ years at 2% compounded quarterly. What is the amount of the discount?

5. How much must be deposited on January 1, 1918, in a bank that pays 3% nominal compounded semiannually, in order to draw out $2000 on January 1, 1925? How much to draw out $2500 on July 1, 1931?

6. A person is offered a piece of property for $2500 cash and $1200 at the end of each year for 4 years. What is the equivalent cash price on a 5½% basis?

7. A person is offered a piece of property for $9800 cash or $6000 cash and $2000 at the end of one year and 2 years. If he can afford to pay cash and can invest money at 6%, should he pay cash?

8. A gave B a note for $350 with interest at 6% nominal converted quarterly, the note being due in 3 years. At the end of a year C bought the note on a 5½% basis. How much did C pay for the note? Did B gain or lose by selling the note?

9. At the birth of his son a father deposited in a bank paying 3½% annually a sum such that he could draw out $600 on the son's 17th, 18th, 19th, and 20th birthdays to use toward the son's education. How much did he deposit?

10. Of 85,441 persons living at age 30 there will be, according to the American Experience Table of Mortality, 57,917 living at age 60. What equal payment by each member of the original group will provide for a payment of $500 to each survivor at age 60, if the funds are invested at 3%?

CHAPTER III

EQUATION OF VALUE AND EQUATION OF PAYMENT

21. Equation of value. The problems so far considered have been solved by direct application of the interest or discount formulas. Often situations are a little more complex, because of the existence of various obligations due at different times, and because it is desired to alter a plan of payment in a way that is fair to both parties involved. In such cases an equation of value is set up. In writing an equation of value all sums involved are accumulated to, or discounted to, a common date, this date being arbitrary in principle, but so chosen in practice as to facilitate computation. Proper consideration of the data then enables one to write an equation from which the unknown quantity can be obtained.

Example. A owes B (a) \$2000 due in 4 years without interest, (b) \$300 due in 3 years with interest at 5% nominal converted semiannually A desires to discharge his obligation in 3 equal yearly payments, the first payment to be made in one year; B agrees to the plan provided a rate of 4% is used in the calculation. Find the amount of the yearly payments.

Let X be the required amount. Take 3 years hence as the time for which all sums are evaluated. We then have the equation

$$X(1.04)^2 + X(1.04) + X = \$300(1.025)^6 + \$2000(1.04)^{-1}.$$

Using the tables,

$$X(1.08160 + 1.04 + 1) = \$300(1.15969) + \$2000(.96154),$$

or

$$3.12160X = \$2270.99,$$

giving

$$X = \$727.51.$$

If the present is chosen for writing the equation of value, we have

$$X(1.04)^{-1} + X(1.04)^{-2} + X(1.04)^{-3}$$
$$= \$300(1.025)^6(1.04)^{-3} + \$2000(1.04)^{-4}.$$

This equation is obtained from the former equation by multiplying by $(1.04)^{-3}$, and will yield the same result as before. But it involves a lengthier computation. The illustration should show the student that in general the particular time chosen for writing the equation of value is arbitrary.

22. Equation of time. Suppose that there are several obligations that are to fall due at different times, and that it is desired to make a single payment equal in amount to the sum of the amounts that are to be paid. We then have to determine when such a payment should be made, the proper time evidently being between the first and the last dates for the different sums considered.

A little algebra is required in the solution of the problem. Let the amounts to be paid be $S_1, S_2, \cdots, S_r$ (these being actual redemption values, interest being included in case there is a note that mentions a sum plus accrued interest), and let them be due in $n_1, n_2, \cdots, n_r$ years, respectively. Let i be the rate of interest to be used in the computation, and let, as before,

$$v = \frac{1}{1+i}.$$

Let n represent the time, in years, when the single payment is to be made.

Taking the present as the time for writing the equation of value, we have

$$(1)\quad v^n(S_1 + S_2 + \cdots + S_r) = v^{n_1}S_1 + v^{n_2}S_2 + \cdots + v^{n_r}S_r.$$

Taking logarithms of both sides and solving for n, we have

$$(2)\quad n = \frac{\log(v^{n_1}S_1 + \cdots + v^{n_r}S_r) - \log(S_1 + \cdots + S_r)}{\log v}.$$

But

$$\log v = -\log(1+i),$$

so that

$$(3)\quad \boldsymbol{n = \frac{\log(S_1 + \cdots + S_r) - \log(v^{n_1}S_1 + \cdots + v^{n_r}S_r)}{\log(1+i)}}.$$

23. Approximate value for the equated time. We have

$$v^n = \frac{1}{(1+i)^n} = (1+i)^{-n}.$$

Now the binomial theorem as given in (6), page 28, is valid for negative values of the exponent (see Appendix I). Thus we have

$$(4)\quad (1+i)^{-n} = 1 - ni + \frac{(-n)(-n-1)}{1\cdot 2}i^2 + \cdots.$$

Unless n is quite large, a fairly accurate result will be given if we use only two terms on the right in (4), since i is small. Thus we take

$$(1 + i)^{-n} = 1 - ni, \text{ approximately.}$$

Similarly

$$\begin{aligned} v^{n_1} &= 1 - n_1 i, \text{ approximately,} \\ v^{n_2} &= 1 - n_2 i, \text{ approximately,} \\ &\cdots\cdots\cdots\cdots \\ v^{n_r} &= 1 - n_r i, \text{ approximately.} \end{aligned}$$

Substituting in (1), we have

$$\begin{aligned} &(1 - ni)(S_1 + S_2 + \cdots + S_r) \\ &\qquad = (1 - n_1 i)S_1 + (1 - n_2 i)S_2 + \cdots + (1 - n_r i)S_r, \end{aligned}$$

from which we find

$$(5) \qquad n = \frac{n_1 S_1 + n_2 S_2 + \cdots + n_r S_r}{S_1 + S_2 + \cdots + S_r}.$$

The interest rate is not present in the approximate formula.* The great convenience of the latter is clear. There is, of course no way to tell easily how nearly the approximate value given by (5) will agree with the correct value as given by (3).

Example. Find on a 5% basis the equated time for the following payments: (a) \$800 due in 1 year, (b) \$1200 due in 2 years, (c) \$1600 due in 3 years, (d) \$2000 due in 4 years. Also use the approximate formula.

The computation according to (3) is as follows:

$$\begin{array}{rcrr} 800v &=& 761.905 & 800 \\ 1200v^2 &=& 1088.435 & 1200 \\ 1600v^3 &=& 1382.141 & 1600 \\ 2000v^4 &=& 1645.405 & 2000 \\ \hline \end{array}$$

$$\log 4877.89 = 3\ 68\ 8232 \qquad \log 5600 = 3.74\ 8188$$

$$\log (1.05) = .02\ 1189$$

$$n = \frac{3.74\ 8188 - 3.68\ 8232}{.02\ 1189} = \frac{.05\ 9956}{.02\ 1189} = 2.83.$$

The approximate formula gives

$$n = \frac{800 + 2(1200) + 3(1600) + 4(2000)}{5600} = \frac{16000}{5600} = 2.86.$$

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* The fact that we have arrived at (5) by assuming i so small that the square and higher powers can be neglected will lead one to expect that (5) is the value for $i = 0$. As a matter of fact, if we put $i = 0$ in (3) we have the *indeterminate* result $n = 0/0$. The student familiar with the subject of indeterminate forms may profitably prove that the value of n in (5) is indeed the *limit* of the value in (3) as i approaches 0.

As stated above, the value given by the approximate formula is greater than that from the exact formula. The difference here is about one part in 100, and would be taken as 11 days.

EXERCISES

1. A man is to make payments of \$1000, \$2000, and \$3000 in 1, 2, and 3 years respectively, as payments on a piece of property. What single payment made in 2 years would be equivalent if money is worth 4% compounded semiannually?

2. A man is to pay \$100 at the end of each month for 6 months. What single payment in 4 months would be equivalent on a 6% basis?

3. As payments on a piece of property, \$10,000 is due in 2 years, and \$10,000 in 4 years. Find the amounts of 4 equal yearly payments, beginning one year hence, that will be equivalent on a 4½% basis.

4. It is desired to pay sums of amount X, $2X$, $3X$ in 2, 4, and 6 years, respectively, in place of a single payment of \$15,000 that is due in 5 years. Find X if money is worth 6%.

5. A offers B a piece of property for \$12,000. B cannot pay cash but offers \$5000 down and an amount X in one year, and $2X$ in 2 years, and agrees to an interest rate of 5% compounded semiannually. A accepts the proposition. Find the value of X.

6. An automobile whose cash price is \$1000 can be purchased by a cash payment of \$350 and payments of the same amount in 6 months and one year. If a person can borrow money from a bank at 3% for 6 months, should he borrow and pay cash for the car?

SOLUTION: In order to compare the two methods it is necessary to pay the bank in two equal payments for the amount that would be borrowed. Let X be the amount of each payment to the bank. Show that

$$X(v + v^2) = 650.$$

Find X and compare with 350.

7. If \$400 is due in 30 days, \$300 in 90 days, and \$550 in 180 days, when can the total \$1250 be paid on a 6% basis? Check by the approximate rule.

8. A has due him \$3000 at the end of each year for 6 years, the first payment to be received in 3 years. When should he be willing to accept a single payment of \$18,000, if money is worth 6%? Solve also by the approximate rule.

CHAPTER IV

ANNUITIES

24. Definition. There are many situations in which a series of equal payments at regular intervals are made. Such a series of payments is called an *annuity*. The word *annuity* itself implies that the payments are made at intervals of a year. In practice, however, the word is used regardless of the length of the period. Thus we can have an annuity with payments made monthly, quarterly, or semiannually, as well as yearly.

25. Annuity with simple interest. In by far the majority of cases the payments in an annuity extend over a period of more than a year, and interest is computed by the compound interest formula. There are, however, cases where the payments may be made monthly, and over a year or less, and simple interest would be used.

Suppose that the amount 1* is deposited in a bank on the first day of each month, and that the savings draw interest at rate i. What will be the value of the account at the end of the year?

Observing that the first deposit draws interest for an entire year, the second for 11 months, the third for 10 months, and so forth, and that the last deposit draws interest for only 1 month, we see that the accumulated amount is

$$S = (1 + i) + \left(1 + \frac{11}{12}i\right) + \left(1 + \frac{10}{12}i\right) + \cdots + \left(1 + \frac{1}{12}i\right)$$

$$= 12 + i\left(1 + \frac{11}{12} + \frac{10}{12} + \cdots + \frac{1}{12}\right)$$

$$= 12 + \frac{i}{12}(1 + 2 + 3 + \cdots + 12).$$

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* It is convenient in what follows to consider all the payments in an annuity as being of amount 1, and not to specify the particular monetary unit.

Using the formula for an arithmetic progression,* we have

$$1 + 2 + 3 + \cdots + 12 = \frac{12}{2}(1 + 12) = 78.$$

Hence

$$S = 12 + \frac{13}{2}i.$$

The work that has been done could be made more general by considering n payments rather than the special number 12, and also intervals other than a month. But not a great deal of use will be made of problems such as the one described, and the student should be able to handle any cases he may meet by working in a way similar to that above.

Example. A man deposited \$50 on the first of each month in a bank that paid 4%. How much was in his account on the last day of the year?

The sum in question was

$$\$50\left(12 + \frac{13}{2} \times .04\right) = \$50(12.26) = \$613.$$

26. Annuity at compound interest. We come now to the usual case of an annuity, one where payments extend over a period of years. To make the matter clear we begin with an example.

Example. Suppose that on January 1, 1920, a man deposited \$200 in a bank paying 4% interest, and that he continued to make such deposits on the first of each year up to and including January 1, 1925. What was the value of his account just after the last deposit?

The accumulated amounts of the various deposits by (1), page 15, were as follows:

First deposit,	$\$200(1.04)^5 =$	\$ 243.331
Second deposit,	$200(1.04)^4 =$	233.972
Third deposit,	$200(1.04)^3 =$	224.973
Fourth deposit,	$200(1.04)^2 =$	216.320
Fifth deposit,	$200(1.04) =$	208.000
Last deposit,	$200 =$	200.000
Total account	=	\$1326.60

It is apparent that the direct method used in the calculation becomes long in case many payments are used. By studying the question in a general way and using a little elementary algebra, a convenient formula is obtained by means of which the accumulated amount of an annuity can be calculated rapidly.

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* See Appendix II.

27. The accumulated amount $s_{\overline{n}|}$. Let the annuity consist of n payments of 1 each, made at intervals of a year, and let the rate be i. Denote the accumulated amount just after the last payment has been made by $s_{\overline{n}|}$ (read "s angle n"). Then noting that the last payment draws no interest, that the next to the last payment draws interest for one year, and so on, and finally that the first payment draws interest for $n - 1$ years, we have

$$(1) \quad s_{\overline{n}|} = 1 + (1 + i) + (1 + i)^2 + \cdots + (1 + i)^{n-1}.$$

We have here a geometric progression of n terms, with first term 1 and ratio $(1 + i)$. Using the formula for the sum,* we have

$$(2) \qquad s_{\overline{n}|} = \frac{(1 + i)^n - 1}{(1 + i) - 1};$$

that is, †

$$(3) \qquad s_{\overline{n}|} = \frac{(1 + i)^n - 1}{i}.$$

Example. Solve the problem of the last section by (3).
We have, since $n = 6$,

$$s_{\overline{6}|}(4\%) = \frac{(1.04)^6 - 1}{.04} = \frac{1.265\,3190 - 1}{.04} = 6.63297.$$

The answer in question is seen to be

$$\$200 \cdot s_{\overline{6}|}(4\%) = \$200 \times 6.63297 = \$1326.59.$$

It is to be carefully observed that the last payment, the nth, is included, and so is taken into account in the formula for

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* Appendix III.

† The symbol $s_{\overline{n}|}$ does not show the interest rate. In particular cases the rate must generally be shown. If the rate is 6% and $n = 10$, we can write $s_{\overline{10}|}$ (6%) or $s_{\overline{10}|\,06}$. In the first form of writing the per cent sign is sufficient to indicate that multiplication is not intended by the quantity in parentheses. In case the symbol occurs as one factor in a long expression, it is often better to use the second form of writing.

$s_{\overline{n}|}$. It is sometimes said that the payments are made at the *ends of the years*. One can think of it in this way if he wishes, but it is to be observed that payments may not coincide with the calendar year, and that any day is no more the end of a year than it is the beginning of a year. The important thing to keep in mind is what is stated above, namely, that the last payment has just been made and is included in $s_{\overline{n}|}$, though of course without interest.

28. Tables for $s_{\overline{n}|}$. Clearly it is a simple matter to construct a table for $s_{\overline{n}|}$ from a table for $(1 + i)^n$. It should be observed, however, that on account of the presence of i in the denominator of (3) it is necessary to have $(1 + i)^n$ to more decimal places than one desires to have in $s_{\overline{n}|}$. Thus we cannot obtain a table to seven places of decimals for $s_{\overline{n}|}$ out of a table to seven places of decimals for $(1 + i)^n$.

Table V, giving values of $s_{\overline{n}|}$, will suffice for solving the pertinent problems in this book, with a very few exceptions.

Let the annual payment be R, and let K be the accumulated amount. Then

$$K = R \cdot s_{\overline{n}|}. \tag{4}$$

If the payments are not round numbers so that tedious arithmetic is needed, it is better to use logarithms. The logarithms of $s_{\overline{n}|}$ are given on the right-hand pages in Table V.

Example 1. A man made regular yearly deposits of \$500 in a bank that paid $3\frac{1}{2}\%$ interest. What was the amount of his account just after the seventh deposit?

$$K = \$500 \cdot s_{\overline{7}|}(3\tfrac{1}{2}\%) = \$500 \times 7.779\,4075 = \$3889.70.$$

Example 2. Find the accumulated amount of 12 payments of \$1230 each on a 4% basis.

$$\begin{aligned} K &= \$1230 \cdot s_{\overline{12}|}(4\%) = \$18{,}482. \\ \log 1230 &= 3.08\,9905 \\ \log s_{\overline{12}|} &= 1.17\,6838 \text{ (directly from table)} \\ \hline \log K &= 4.26\,6743 \end{aligned}$$

If the interest rate or the number of payments is not provided for in the table, one must resort to direct calculation by (3), using logarithms.

Example 3. Find $s_{\overline{24}|}(1\frac{1}{4}\%)$.

From (3),

$$s_{\overline{24}|}(1\tfrac{1}{4}\%) = \frac{(1.0125)^{24} - 1}{.0125}.$$

The calculation follows:

$$\begin{aligned} \log 1\ 0125 &= 0\ 00\ 5395 \\ 24 \log 1.0125 &= 0\ 12\ 948 \\ (1.0125)^{24} - 1 &= 0.34\ 734 \\ 1/5\cdot[(1.0125)^{24} - 1] &= 0.06\ 947 \\ s_{\overline{24}|}(1\tfrac{1}{4}\%) &= 27.788. \end{aligned}$$

NOTE. Instead of using logarithms to divide by .0125, the numerator was multiplied by 400/5, which could be done easily; the division by 5 was made first. After multiplying log 1.0125 by 24 we are entitled to only 5 decimal places in the result, for plainly the 7th decimal in log 1.0125 would influence the 6th decimal in 24 log 1.0125 very decidedly. It is thus seen that we should have log $(1 + i)$ to at least 7 decimals if we are to evaluate $s_{\overline{n}|}$ to 6 figures, and may need it to more places if n is large.

29. Nominal rate. In obtaining $s_{\overline{n}|}$ we assumed that there was one payment a year, and that the yearly rate was i. It is obvious, however, that the essential thing is that there be n payments, that i be the rate per conversion period, and that each payment be of amount 1. With this in mind it is easy to find the amount of an annuity when payments are made more often than once a year, and the appropriate nominal rate is given.

Example. Payments on an annuity of $200 each are made twice a year, and are accumulated at a nominal rate of 5% compounded semiannually. What is the amount at the end of 8 years?

Here there are 16 payments, and the rate per conversion period is $2\frac{1}{2}\%$. Hence

$$K = \$200\cdot s_{\overline{16}|}(2\tfrac{1}{2}\%) = \$200 \times 19.38022 = \$3876.04.$$

It is possible to write down a special formula to provide for a nominal interest rate, but it is better to employ the basic formula where the adaptation is so simple.

EXERCISES

A

1. On the first of each month a man made a deposit of $60 in a bank that paid 2%. What was the amount of his account just after he made the eighth deposit? Use simple interest

2. Beginning on his 21st birthday a man made yearly deposits of \$250 in a savings account that paid 3%. What was the value of his account just after his 30th birthday?

3. To provide for replacements, a company made semiannual deposits of \$1500 each in a fund that paid 4% nominal, converted semiannually. How much was in the fund at the end of 10 years, assuming the deposits made at the end of the 6-months periods?

4. If \$15 is deposited at the end of each quarter in an account that pays 4% nominal, converted quarterly, how much will be in the account at the end of 15 years?

5. On January 1, 1915, a man bought a lot for \$2000. He paid \$25 taxes semiannually, in the first weeks of January and July respectively, the first payment being made 6 months after he bought the lot. If he estimated his money conservatively as worth 4% nominal, converted semiannually, how much must he sell the lot for in January, 1925, in order not to lose money, assuming that he paid the January taxes?

6. A company borrowed \$100,000 at 6%, the agreement providing that all unpaid interest payments be accumulated at 7%. The company failed to make five consecutive interest payments, but cleared up all arrears and made the next payment, when the latter was due. What was the total amount paid?

7. A man made deposits of \$200 each on April 10 and October 10 of each year in a bank paying 3½% nominal converted semiannually, the first deposit being made April 10, 1912. He withdrew his savings October 9, 1924. How much did he withdraw?

8. The principal \$2345 drew interest at 5% nominal, payable quarterly. The interest was deposited in a savings account paying 2% nominal, converted quarterly. How much was in the account at the end of 8 years?

9. A man deposited \$25 at the end of each month in an account paying 3%, with simple interest for fractions of a year. At the end of each 6-months period the amount in the account was transferred to a second account paying 4% nominal converted semiannually. How much was in the latter account at the end of 15 years?

10. Find $s_{\overline{75}|}(1\%)$ by using logarithms.

11. Find $s_{\overline{25}|}(4.35\%)$ by (3).
Use $\log 1.0435 = 0.018\,4925$.

B

1. On the 10th of each month a man made a deposit of \$50 in a bank that paid 2½%. What was the amount of his account after he made the eighth deposit? Use simple interest.

2. Beginning on his 25th birthday a man made yearly deposits of \$425 in a savings account that paid 3½%. What was the value of his account just after his 35th birthday?

3. To provide for replacements a company made quarterly deposits of \$800 each in a fund that paid 4% nominal converted quarterly. How much was in the fund at the end of 6 years, assuming that the deposits were made at the end of the 3-months periods?

4. If \$20 is deposited at the end of every second month in an account that pays 3% nominal converted every two months, how much will be in the account at the end of 10 years?

5. On April 1, 1920, a man bought a lot for \$3000. He paid \$35 taxes semiannually in the first weeks of April and October, respectively, the first payment being made one year after he bought the lot. If he estimated his money conservatively as worth 5% nominal converted semiannually, how much must he sell the lot for in the first week of November, 1930, if he is not to lose money, assuming that he makes the tax payment then due?

6. A company borrowed \$125,000 at 5%, the agreement providing that all unpaid interest payments be accumulated at 6%. The company defaulted three consecutive interest payments, but made the next payment and paid all arrears at the same time What was the total amount paid?

7. A man made deposits of \$250 each on February 20 and August 20 of each year in a bank paying 4% nominal converted semiannually, the first deposit being made February 20, 1920. He withdrew his savings August 19, 1928. How much did he withdraw?

8. The principal \$5250 earned 3% interest semiannually. The interest was deposited in a savings account paying 5% converted semiannually. How much was in the savings account at the end of 15 years?

9. A man deposited \$40 at the end of each month in an account paying $3\frac{1}{2}\%$ At the end of each 6-months period the amount in the account was transferred to a second account paying 5% nominal converted semiannually. How much was in the latter account at the end of 8 years?

10. Find $s_{\overline{68}|}(1\frac{1}{2}\%)$, using logarithms.

11. Find $s_{\overline{18}|}$ (5.12%) by (3)

Use log 1.0512 = 0.021 6854.

30. The present value $a_{\overline{n}|}$. We again begin with a special case.

Example 1. How much must be deposited in a bank that pays 2% in order that 6 yearly withdrawals of \$200 each can be made, the first withdrawal being made in one year?

It is clear that the amount to be deposited is merely the sum of the present values of the separate withdrawals. By (16), page 33, the present values are:

First withdrawal,	$\$200v\ (2\%)$	=	\$ 196.078
Second withdrawal,	$200v^2$	=	192.234
Third withdrawal,	$200v^3$	=	188.464
Fourth withdrawal,	$200v^4$	=	184.770
Fifth withdrawal,	$200v^5$	=	181.146
Sixth withdrawal,	$200v^6$	=	177.594
Deposit required,			\$1120.29

The payments in the example constitute an annuity, of which the present value is $1120.29.

Such a direct method, long even for so few payments as six, becomes forbiddingly crude when many payments are involved. As in the case of $s_{\overline{n|}}$ the use of a little algebra enables us to simplify the problem greatly.

Let there be n payments each of amount 1, the first payment being due in one year. Denote the present value of the entire annuity by $a_{\overline{n|}}$ (read "a angle n"). Let, as usual,

$$v = \frac{1}{1+i}.$$

We have n different payments, of which the respective present values are $v, v^2, v^3, \cdots, v^n$. For the present value of all payments we then have

$$a_{\overline{n|}} = v + v^2 + \cdots + v^n. \tag{5}$$

Using the formula for the sum of a geometric progression, we have

$$a_{\overline{n|}} = \frac{v(1-v^n)}{1-v} = \frac{1-v^n}{\frac{1}{v}-1}. \tag{6}$$

Replacing v by its value in terms of i, we have

$$\boldsymbol{a_{\overline{n|}} = \frac{1-(1+i)^{-n}}{i}}. \tag{7}$$

A very convenient mixed form involving both v and i is

$$\boldsymbol{a_{\overline{n|}} = \frac{1-v^n}{i}}. \tag{8}$$

From the table of values of v^n we can easily obtain one for $a_{\overline{n|}}$, recalling the remarks about the number of decimal places made in connection with $s_{\overline{n|}}$.

Example 2. Find $a_{\overline{6|}}(2\%)$.

We have

$$a_{\overline{6|}}(2\%) = \frac{1-v^6(2\%)}{.02} = \frac{1-.887\,9714}{.02}$$

$$= \frac{.112\,0286}{.02} = 5.60143.$$

Table VI is a table for $a_{\overline{n}|}$, the quantity itself being given on left-hand pages, and the logarithm on right-hand pages.

Let R be the annuity payment; then the present value is

$$A = R \cdot a_{\overline{n}|}. \tag{9}$$

Example 3. The present value of an annuity of \$200 under conditions considered above is, making use of the value just found,

$$\$200 \cdot a_{\overline{6}|}(2\%) = \$200 \times 5.60143 = \$1120.29.$$

Again it is to be recalled that the payments of the annuity are to be made at the ends of the years. That is, the first payment is due in *one year from the time* for which the present value is calculated.

When the payments of the annuity are made more than once a year, and the appropriate nominal rate of interest is given, we can again use (9), in a way similar to the procedure when the accumulated amount was considered.

Example 4. The present value of quarterly payments of \$100 for 8 years, first payment due in 3 months, at 6% nominal, compounded quarterly, is

$$\$100 \cdot a_{\overline{32}|}(1\tfrac{1}{2}\%) = \$2526.71.$$

31. Annuity schedule. It is important to see how the present value of an annuity provides for the different payments, when interest is taken on the entire capital invested at the beginning of each year.

In the example above it was found that the present value of an annuity of \$200 a year for 6 years was \$1120.29. The schedule that shows the yearly interest and withdrawals follows.

YEAR	PRINCIPAL AT BEGINNING OF YEAR	INTEREST	TOTAL AT END OF YEAR	WITH-DRAWAL	BALANCE
1st	\$1120.29	\$22.41	\$1142.70	\$200	\$942.70
2nd	942.70	18.85	961.55	200	761.55
3rd	761.55	15.23	776.78	200	576.78
4th	576.78	11.53	588.31	200	388.31
5th	388.31	7.77	396.08	200	196.08
6th	196.08	3.92	200.00	200	0.00

The procedure by which $a_{\overline{n}|}$ was obtained was not in conformity with such a schedule. We considered, rather, the

The payments in the example constitute an annuity, of which the present value is $1120.29.

Such a direct method, long even for so few payments as six, becomes forbiddingly crude when many payments are involved. As in the case of $s_{\overline{n|}}$ the use of a little algebra enables us to simplify the problem greatly.

Let there be n payments each of amount 1, the first payment being due in one year. Denote the present value of the entire annuity by $a_{\overline{n|}}$ (read "a angle n"). Let, as usual,

$$v = \frac{1}{1+i}.$$

We have n different payments, of which the respective present values are v, v^2, v^3, $\cdots$, v^n. For the present value of all payments we then have

$$a_{\overline{n|}} = v + v^2 + \cdots + v^n. \tag{5}$$

Using the formula for the sum of a geometric progression, we have

$$a_{\overline{n|}} = \frac{v(1-v^n)}{1-v} = \frac{1-v^n}{\frac{1}{v}-1}. \tag{6}$$

Replacing v by its value in terms of i, we have

$$a_{\overline{n|}} = \frac{1-(1+i)^{-n}}{i}. \tag{7}$$

A very convenient mixed form involving both v and i is

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From the table of values of v^n we can easily obtain one for $a_{\overline{n|}}$, recalling the remarks about the number of decimal places made in connection with $s_{\overline{n|}}$.

Example 2. Find $a_{\overline{6|}}(2\%)$.

We have

$$a_{\overline{6|}}(2\%) = \frac{1-v^6(2\%)}{.02} = \frac{1-.887\,9714}{.02}$$

$$= \frac{.112\,0286}{.02} = 5.60143.$$

Table VI is a table for $a_{\overline{n}|}$, the quantity itself being given on left-hand pages, and the logarithm on right-hand pages.

Let R be the annuity payment; then the present value is

$$A = R \cdot a_{\overline{n}|}. \tag{9}$$

Example 3. The present value of an annuity of \$200 under conditions considered above is, making use of the value just found,

$$\$200 \cdot a_{\overline{6}|}(2\%) = \$200 \times 5.60143 = \$1120.29.$$

Again it is to be recalled that the payments of the annuity are to be made at the ends of the years. That is, the first payment is due in *one year from the time* for which the present value is calculated.

When the payments of the annuity are made more than once a year, and the appropriate nominal rate of interest is given, we can again use (9), in a way similar to the procedure when the accumulated amount was considered.

Example 4. The present value of quarterly payments of \$100 for 8 years, first payment due in 3 months, at 6% nominal, compounded quarterly, is

$$\$100 \cdot a_{\overline{32}|}(1\tfrac{1}{2}\%) = \$2526.71.$$

31. Annuity schedule. It is important to see how the present value of an annuity provides for the different payments, when interest is taken on the entire capital invested at the beginning of each year.

In the example above it was found that the present value of an annuity of \$200 a year for 6 years was \$1120.29. The schedule that shows the yearly interest and withdrawals follows.

YEAR	PRINCIPAL AT BEGINNING OF YEAR	INTEREST	TOTAL AT END OF YEAR	WITH-DRAWAL	BALANCE
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2nd	942.70	18.85	961.55	200	761.55
3rd	761.55	15.23	776.78	200	576.78
4th	576.78	11.53	588.31	200	388.31
5th	388.31	7.77	396.08	200	196.08
6th	196.08	3.92	200.00	200	0.00

The procedure by which $a_{\overline{n}|}$ was obtained was not in conformity with such a schedule. We considered, rather, the amounts that compounded from the beginning would provide for each separate payment, and then took the sum of these different amounts.

32. Other derivations of $s_{\overline{n|}}$ and $a_{\overline{n|}}$. We have derived the formulas for $s_{\overline{n|}}$ and $a_{\overline{n|}}$ in what may be called the most natural way, employing for the purpose the formula for the sum of a geometric progression.

It is possible to derive the formulas otherwise, by using equations of value and resorting to reasoning that it is highly important for the student to understand if he is to attain a mastery of the subject.

Suppose that the amount 1 is put at compound interest for n years. We know that it will accumulate to $(1 + i)^n$. On the other hand, suppose that the interest earned at the end of the first year, namely i, is put into a separate account, which also draws interest at rate i. The original principal, namely 1, will again earn the interest i; let this be put into the second account, and so on, until the end of the n years. The accumulated amount of the second account, into which each year the amount i has been put, will be $i \cdot s_{\overline{n|}}$, from the definition of $s_{\overline{n|}}$. There will also be the original principal 1. Now the two procedures, that of thinking of the original principal as compounding and that of segregating the interest into a second account, must evidently lead to the same final amount. Hence

$$i \cdot s_{\overline{n|}} + 1 = (1 + i)^n, \tag{10}$$

or, solving for $s_{\overline{n|}}$,

$$s_{\overline{n|}} = \frac{(1 + i)^n - 1}{i}.$$

More space has been used than in the first derivation, but this was because an effort was made to explain the reasoning at length. A person proficient in the subject would write equation (10) after very brief reflection.

By considering that one has loaned the amount 1, producing an annuity of i with present value $i \cdot a_{\overline{n|}}$, and that the principal is due for return in n years and thus has a present value of v^n, show that

$$(11) \qquad i \cdot a_{\overline{n}|} + v^n = 1,$$

which gives

$$a_{\overline{n}|} = \frac{1 - v^n}{i},$$

as found above in (8).

Assuming that the present value of an annuity of 1 at rate i is accumulated for n years, we see that

$$(12) \qquad a_{\overline{n}|} \cdot (1 + i)^n = s_{\overline{n}|},$$

a relation which can be verified from (3) and (7).

The student is urged to think over the equations (10), (11), and (12) until the difficulty they may at first possess disappears, and they become natural expressions for him to write.

EXERCISES

A

1. Find $a_{\overline{20}|}(3\%)$ by using Table IV.

2. Find $a_{\overline{14}|}(5\frac{1}{2}\%)$ by using logarithms.

3. A house is bought for $4000 cash and payments every year of $1500 each for 6 years. Find the equivalent cash price on a 5% basis.

4. A house is bought for $2500 cash and monthly payments of $50 for 5 years. What is the equivalent cash price on the basis of 6% nominal, converted monthly?

5. On January 1, 1920, a man deposited $5000 in a bank that paid 4% nominal converted semiannually. He withdrew $400 semiannually, starting July 1, 1920, and continuing until January 1, 1926. What was the value of his account after the last-mentioned withdrawal?

In terms of $a_{\overline{n}|}$ show that the balance is

$$A = [5000 - 400a_{\overline{12}|}](1.02)^{12}.$$

In terms of $s_{\overline{n}|}$ show that

$$A = 5000(1.02)^{12} - 400s_{\overline{12}|}.$$

Compare the two solutions from the standpoint of computation involved.

6. A man takes out an insurance policy on which the annual premium is $128.70. How much would he have to deposit in a bank to provide for the next 19 premiums, if the bank pays 2%? Suppose that the man dies a few days after the end of 8 years. In addition to his insurance, how much would he leave in the bank?

7. A man is offered a piece of property for $10,000 cash. He wishes to make a cash payment, and quarterly payments of $250 for 5 years. What should the cash payment be if the seller discounts future payments at 6% nominal, converted quarterly?

8. Show that

$$a_{\overline{m+n}|} = v^m a_{\overline{n}|} + a_{\overline{m}|},$$

by considering the meaning of the various quantities, and also by using the formulas for the various present value symbols.

Use the equation to find $a_{\overline{56}|}$ (6%) from values in the tables.

9. Show by direct reasoning that

$$s_{\overline{m+n}|} = (1 + i)^n s_{\overline{m}|} + s_{\overline{n}|},$$

by using the expression for the value of $s_{\overline{n}|}$, etc., and also by multiplying the equation in Exercise 8 by $(1 + i)^{m+n}$ and using equation (12).

10. Find $s_{\overline{64}|}(5\%)$ by using the equation in the last exercise.

B

1. Find $a_{\overline{24}|}(2\frac{1}{2}\%)$ by using Table IV.

2. Find $a_{\overline{16}|}(5\frac{1}{4}\%)$ by using logarithms.

3. A house is bought for $2500 cash and payments of $800 a year for 12 years. What is the equivalent cash price on a 6% basis?

4. A house is bought for $3000 cash and quarterly payments of $500 for 4 years. What is the equivalent cash price on the basis of 5% nominal, converted quarterly?

5. On April 10, 1912, a man deposited $4000 in a bank that paid 4% nominal converted semiannually. He withdrew $200 semiannually, starting on October 10, 1912, and continuing until April 10, 1922. What was the value of his account after the last withdrawal?

Discussion in Exercise 5, list A.

6. A man takes out an insurance policy on which the annual premium is $78.60. How much would he have to deposit in a bank to provide for the next 29 payments, if the bank pays $1\frac{3}{4}\%$? Suppose the man dies a few days after the end of 18 years. How much would he leave in the bank?

7. A man is offered a piece of property for $8000. He wishes to make a cash payment and semiannual payments of $400 for 8 years. What should the cash payment be if the seller discounts future payments at 6% nominal converted semiannually?

8. Exercise 8, list A. Use the equation to find $a_{\overline{72}|}(4\%)$.

9. Exercise 9, list A.

10. Find $s_{\overline{85}|}(6\%)$ by using Exercise 9.

33. Annuities due. The following facts are recalled:

The quantity $s_{\overline{n}|}$ gives the accumulated amount of an annuity of 1 per year just after the nth payment has been made.

The quantity $a_{\overline{n}|}$ gives the present value of an annuity of 1 per year, the first payment to be made one year hence.

It is frequently said that the payments on the annuity are made at the *end* of the year.

Most problems in annuities are such that either $s_{\overline{n}|}$ or $a_{\overline{n}|}$ applies to them without any modification.

There are problems, however, that do not come directly under $s_{\overline{n}|}$ or $a_{\overline{n}|}$ without a slight modification, and some common cases will now be considered.

Example 1. On January 1, 1910, a man made a deposit of \$200 in a bank that paid 4% interest; he continued such deposits until January 1, 1924. On January 1, 1925, he withdrew his account. What was the amount of the withdrawal?

There are two methods of solution.

(1) The value of the account just after the deposit of January 1, 1924, was $\$200 s_{\overline{15}|}(4\%)$, so that the amount on January 1, 1925, was

$$\$200 s_{\overline{15}|} \cdot (1.04) = \$200 \times 20.023588 \times 1.04 = \$4164.91.$$

(2) If a deposit had been made on January 1, 1925, the account would have had the value $\$200 s_{\overline{16}|}$. It is clear that the amount in the account before such a deposit was made, which was the amount the man withdrew, was

$$\$200(s_{\overline{16}|} - 1) = \$200 \times 20.82453 = \$4164.91.$$

The second method of working the problem is numerically shorter.

Example 2. The yearly cost of a government 20-payment life policy for \$1000 for a veteran aged 25 was \$22.56. What was the value of the payments on a 3½% basis, assuming that the insured lived to make all payments?

The first premium on an insurance policy must be paid at once. The present value of the future 19 premiums can be found by means of $a_{\overline{19}|}$. The required present value of all premiums was evidently

$$\$22.56[1 + a_{\overline{19}|}(3\tfrac{1}{2}\%)] = \$22.56 \times 14.70984 = \$331.85.$$

The examples that have been worked are cases of *annuities due*, annuities in which the first payment is made at once. For such annuities the symbols $\mathbf{s}_{\overline{n}|}$ and $\mathbf{a}_{\overline{n}|}$ are used.

The quantity $\mathbf{s}_{\overline{n}|}$ is the accumulated amount of an annuity due of 1 per year, at the end of n years.

The quantity $\mathbf{a}_{\overline{n}|}$ is the present value of an annuity due of 1 per year, the last payment being due in $n - 1$ years.

In terms of $\mathbf{s}_{\overline{n}|}$ and $\mathbf{a}_{\overline{n}|}$ we have

$$\mathbf{s}_{\overline{n}|} = (1 + i)s_{\overline{n}|} = s_{\overline{n+1}|} - 1, \tag{13}$$

$$\mathbf{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}. \tag{14}$$

In working problems involving annuities due, it is not necessary to use first the symbol $\mathbf{s}_{\overline{n}|}$ or $\mathbf{a}_{\overline{n}|}$ and then introduce $s_{\overline{n+1}|}$ or $a_{\overline{n-1}|}$ by (13) or (14). One can always analyze the problem directly in terms of $s_{\overline{n}|}$ or $a_{\overline{n}|}$. It is desirable not to have an unnecessary number of symbols, especially when there may be difficulty in distinguishing some of them clearly. But the symbols for annuities due are a conventional part of the subject of finance.

34. Deferred annuities. The payments on an annuity may not begin in a year, as in the basic case, or at once as in the case of an annuity due. They may begin at the end of any number of years.

Example. How much must a man deposit on April 10, 1920, in a bank that pays 4% in order that he can make 10 yearly withdrawals of \$500 each, the first withdrawal being made April 10, 1935?

There are two solutions.

(1) The amount that must be in the bank on April 10, 1934, is $\$500 \cdot a_{\overline{10}|}(4\%)$, according to the meaning of $a_{\overline{10}|}$. This sum need merely be discounted 14 years back to April 10, 1920, so that the required amount is *

$$\begin{aligned}\$500 \cdot a_{\overline{10}|}v^{14}(4\%) &= \$500 \times 8.110\,8958 \times .577\,4751\\ &= \$4055.4479 \times .577\,4751 = \$2341.92.\end{aligned}$$

(2) The amount necessary to provide for withdrawals beginning April 10, 1921, and continuing to April 10, 1944 (the date of the last actual withdrawal), would be $\$500 \cdot a_{\overline{24}|}$. The amount necessary to provide for withdrawals beginning April 10, 1921, and continuing until April 10, 1934 (which are withdrawals not actually made) would be $\$500 \cdot a_{\overline{14}|}$. The amount needed to provide for the withdrawals desired would be the difference between the amounts mentioned. Hence the required amount is

$$\begin{aligned}\$500(a_{\overline{24}|} - a_{\overline{14}|})(4\%) &= \$500(15.246\,963 - 10.563\,123)\\ &= \$500 \times 4.683\,840 = \$2341.92.\end{aligned}$$

The second method is much the simpler from the computational standpoint.

The annuity just considered is an instance of a *deferred annuity*. The annuity is said to be deferred 14 years, although the

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* The expression

$$a_{\overline{10}|}v^{14}$$

has two symbols that do not show the interest rate. We could write

$$a_{\overline{10}|\,04}\,\frac{1}{(1.04)^{14}}.$$

The method of writing employed in the text, though open to some objection, is convenient. Of course it would be entirely unacceptable to write

$$a_{\overline{10}|}v^{14}(.04).$$

first payment is not to be made for 15 years, for the basic formula $a_{\overline{n}|}$ itself provides that the first payment is not made for one year.

An annuity in which the first payment is to be made in $m + 1$ years is said to be *deferred m years*. The present value of 1 per year for n years deferred m years is represented by $_{m|}a_{\overline{n}|}$.

Reasoning as in the example it is seen that

$$_{m|}a_{\overline{n}|} = a_{\overline{n}|}v^m, \tag{15}$$

and also that

$$_{m|}a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|}, \tag{16}$$

the second formula being often preferable for computation.

EXERCISES

1. A man who had made 11 annual payments of \$16.73 each on a life insurance policy for \$1000 died just before the 12th payment was due. How much more did he leave than he would have left if instead of taking out the insurance he had deposited the premium in a bank that paid 3½% interest?

2. A man was paying \$18.50 semiannually on an insurance policy for \$2000 and died 2 months after making the 23rd payment. How much more did he leave than he would have left if he had deposited the premiums in a bank that paid 4% nominal, converted semiannually, using simple interest for the 2-months period?

3. A man deposited \$400 at the beginning of each January and July in a bank that paid 2½% nominal, converted semiannually. How much could he withdraw at the end of 12 years?

4. On the first of each month a man paid \$75 on the purchase price of a house, the house to become his at the 96th payment. What was the equivalent cash price on the basis of 6% nominal, converted monthly?

5. On the birth of a son a father deposited an amount in a bank paying 2½% interest such that he could withdraw \$2000 on the son's 17th, 18th, 19th, and 20th birthdays. How much did he deposit?

6. What is the present value of an annuity with quarterly payments of \$500 each, continuing for 8 years, the first payment to be received in 6 years, if money is worth 5% converted quarterly?

7. On the first of each year from 1912 to 1921 the sum of \$5000 was deposited in a fund paying 3½% interest. No further deposits were made, and the account was withdrawn on January 1, 1930. How much was withdrawn? (There are two solutions, one of which involves little calculation.)

35. The quantities *n* and *i* as unknowns. In formula (4) there are four letters—R, n, i (not explicitly shown, but contained in $s_{\overline{n|}}$), and K. We have so far considered problems in which the values of R, n, and i were given, and the value of K was to be found. A similar remark applies to (9), where A has thus far been considered the unknown. Very often the payment R is the quantity that must be found, the accumulated amount K or the present value A being given. Such problems will be considered in a later chapter. Problems also occur when n or i is the unknown to be found, and such problems will now be considered.

Example 1. How many full payments of $400 each and what final payment must be made into a fund paying 4% in order to accumulate $10,000?

We have

$$400 \cdot s_{\overline{n|}}(4\%) = 10{,}000,$$

which gives

$$s_{\overline{n|}}(4\%) = 25.$$

From the table for $s_{\overline{n|}}(4\%)$ we find

$$s_{\overline{17|}} = 23.697\,5124, \qquad s_{\overline{18|}} = 25.645\,4129.$$

It is seen then that 17 full payments are necessary, and a final payment less than $400. Further, 18 deposits of $400 each would give

$$\$400 \cdot s_{\overline{18|}} = \$10{,}258.17.$$

Thus the final deposit should be

$$\$400 - \$258.17 = \$141.83.$$

An algebraic solution of such a problem is possible, as is seen in Exercise 2 below. The solution by means of a table is simpler, however.

Let us consider an instance where i is the unknown.

Example 2. A man gave a friend $500 a year for 10 years and received a $6500 interest in the friend's business. What rate of interest was allowed on the money advanced?

Here we have

$$500 \cdot s_{\overline{10|}}(\text{rate } i) = 6500,$$

so that

$$s_{\overline{10|}}(\text{rate } i) = 13.$$

From the table we see that i is between .05 and .06, and we have

Rate	$s_{\overline{10}\rceil}$
.05	12.5779
i	13
.06	13.1808

By proportion,

$$\frac{i - .05}{.06 - .05} = \frac{13 - 12.5779}{13.1808 - 12.5779} = \frac{.4221}{.6029} = .7001.$$

Thus

$$i - .05 = .7001 \times .01,$$

so that

$$i = .0570.$$

Accordingly 5.70% interest was earned on the money advanced.

An algebraic solution of this problem would be difficult. See Exercise 2, page 61.

EXERCISES

A

1. It is desired to accumulate \$3000 by saving \$150 semiannually, the fund being accumulated at 3% nominal, compounded every 6 months. How long will be required and what will be the final payment?

2. If savings can be accumulated at 3% nominal, converted monthly, how long will it take to obtain approximately \$500 by saving \$23 a month, put at interest at the end of the month?

3. A city wishes to borrow \$100,000. It is estimated that in addition to paying interest, a sum of \$5000 can be set aside annually to retire the bonds. If such amounts are put in a sinking fund drawing 4% interest, when can the bonds be paid, and what amount is required the last year to complete the fund for retirement?

4. If \$1000 is deposited yearly in a bank that pays 1½%, when will the account amount to \$12,000?

5. If 20 deposits of \$425 each amount to \$12,900, what is the interest rate?

6. One department of a firm advanced \$1000 a year for 5 years to a second department. At the end of 10 years the first department was repaid \$6900. What interest had been paid it for the use of its money?

7. The sum of \$10,000 was deposited in a bank paying 3½% interest, and yearly withdrawals of \$600 were made. Find the number of withdrawals and the final partial withdrawal.

SUGGESTION: Set up the equation

$$600a_{\overline{n}\rceil}(3\tfrac{1}{2}\%) = 10{,}000,$$

so that

$$a_{\overline{n}\rceil}(3\tfrac{1}{2}\%) = 16.67.$$

Show from the table that 25 full withdrawals were made, that

$$\$600a_{\overline{25}|} = \$9888.90$$

was required to provide for them, and thus that $111.10 by accumulation produced the 26th withdrawal. Find the final withdrawal.

8. Show that the 26th withdrawal in Exercise 7 can be computed readily from

$$\$10{,}000(1 + i)^{26} - \$600(s_{\overline{26}|} - 1).$$

9. A man deposited $4000 in a bank paying 4% nominal, converted semiannually; he withdrew $250 each 6 months, and made a final withdrawal of less than $250. Find the number of full withdrawals and the last partial withdrawal.

10. A house is offered for sale for $10,000 cash. A person offers $3500 cash and payments of $800 a year. How many payments should the seller ask for on a 6% basis if he is not to receive less than the equivalent of his cash offer?

11. An annuity of $400 a year for 15 years is offered for $4200. What interest rate is being used?

12. Compare an annuity of $500 a year for 10 years offered for $3800 and one for $1000 for 13 years offered for $9700.

B

1. It is desired to accumulate $8000 by saving $250 semiannually, the fund being accumulated at 3% nominal, compounded every 6 months. How long will be required and what will be the final payment?

2. If savings can be accumulated at 6% nominal converted monthly, how long will it take to obtain approximately $3000 by saving $50 a month, put at interest at the end of the month?

3. A city wishes to borrow $200,000. It is estimated that in addition to paying interest, a sum of $6000 can be set aside annually to retire the bonds. If such amounts are put in a sinking fund drawing 2½% interest, when can the bonds be paid, and what amount is required the last year to complete the fund for retirement?

4. If $500 is deposited yearly in a bank that pays 1¾%, when will the account amount to $11,000?

5. If 15 deposits of $525 each amount to $11,000, what is the interest rate?

6. One department of a firm advanced another department $2000 for 7 years, and at the end of 10 years $18,000 was returned. What rate of interest had been allowed for the use of the money?

7 and **8.** Same as in list A.

9. A man deposited $10,000 in a bank paying 4% nominal converted quarterly; he withdrew $350 each quarter, and a final withdrawal less than $350. Find the number of full withdrawals and the last partial withdrawal.

10. A house is offered for sale for \$8500 cash. A man offers \$3000 cash and yearly payments of \$750. How many payments should the seller require on a 7% basis if he is not to receive less than the equivalent of his cash offer?

11. An annuity of \$420 a year for 10 years is offered for \$3200. What is the interest rate?

12. Compare an annuity of \$500 a year for 15 years offered for \$3200 and one of \$1000 a year for 20 years offered for \$5000.

ALGEBRAIC EXERCISES

1. Show from (3) and (4) that

$$n = \frac{\log (Ki + R) - \log R}{\log (1 + i)}.$$

What interpretation should be made if n is found to be nonintegral, as is likely to be the case?

2. Show that if K, R, and n are given, and i is the unknown, the algebraic determination of i involves solving an equation of degree $n - 1$. [Recall that by the binominal theorem $(1 + i)^n = 1 + ni + \cdots$.]

If $n = 2$, show that

$$i = \frac{K}{R} - 2.$$

If $n = 3$, show that we have the quadratic equation

$$i^2 + 3i - \frac{K}{R} + 3 = 0.$$

3. From (7) and (9) show that

$$n = \frac{\log R - \log (R - iA)}{\log (1 + i)}.$$

4. Show that in the case of present value the determination of i when A, R, and n are given involves solving an equation of degree n.

If $n = 2$, show that we have the quadratic

$$i^2 + \left(2 - \frac{R}{A}\right) i + 1 - 2\frac{R}{A} = 0.$$

CHAPTER V

ANNUITIES *(Continued)*

Different Conversion and Payment Periods

36. Conversion more frequent than payments. There are instances where interest is converted more often than payments are made on an annuity. An example will be examined in detail.

What is the accumulated amount of 1 per year for 10 years, if interest is paid twice a year on a 4% nominal basis?

Observing that the different payments draw interest for 18, 16, 14, $\cdots$, 4, 2, 0 conversion periods, it is seen that the accumulated amount is

$$1 + (1.02)^2 + (1.02)^4 + \cdots + (1.02)^{18}.$$

This geometric progression of 10 terms with ratio $(1.02)^2$ has the sum

$$\frac{(1.02^2)^{10} - 1}{(1.02)^2 - 1} = \frac{(1.02)^{20} - 1}{(1.02)^2 - 1}.$$

The values of $(1.02)^{20}$ and $(1.02)^2$ can be taken from the table for $(1 + i)^n$, and the above fraction calculated.

For purposes of interpretation a transformation will be made. We observe that $(1.02)^2 = 1.0404$, so that the quantity above can be written

$$\frac{(1 + .0404)^{10} - 1}{.0404}.$$

Comparing this with (3), page 45, we see that the result is merely $s_{\overline{10}|}$ for an interest rate of 4.04%. The latter is the effective rate that corresponds to a nominal rate of 4% compounded semiannually. A little reflection will show that the answer to the problem should indeed be $s_{\overline{10}|}$ (4.04%). The use of the geometric series is not necessary, but it is worth while for the student to understand the solution of the problem by such a means. We assume then that the new problem can always be solved by using $s_{\overline{n}|}$ evaluated for the effective rate that corresponds to the given nominal rate.

For convenience of reference we shall use ${}_ps_{\overline{n}|}$ for the accumulated amount of an annuity of 1 a year for n years, interest being converted p times a year, though this is not a standard symbol.

Let i be the effective rate corresponding to a nominal rate j converted p times a year; then by (3), page 45,

$$(1) \qquad {}_ps_{\overline{n}|} = \frac{(1+i)^n - 1}{i}.$$

A very notable transformation can be made by introducing $1 + i$ and i in terms of $j_{(p)}$ and p. By (2), page 26,

$$1 + i = \left(1 + \frac{j_{(p)}}{p}\right)^p, \quad i = \left(1 + \frac{j_{(p)}}{p}\right)^p - 1.$$

When these values are substituted in (1) and the numerator and the denominator of the resulting fraction are both divided by $j_{(p)}/p$, we obtain

$$(2) \qquad {}_ps_{\overline{n}|} = \frac{\dfrac{\left(1 + \dfrac{j_{(p)}}{p}\right)^{pn} - 1}{\dfrac{j_{(p)}}{p}}}{\dfrac{\left(1 + \dfrac{j_{(p)}}{p}\right)^{p} - 1}{\dfrac{j_{(p)}}{p}}}.$$

The numerator of the complex fraction on the right is $s_{\overline{np}|}$ and the denominator is $s_{\overline{p}|}$, both at rate $j_{(p)}/p$, as is seen by referring to (3), page 45.

Consequently

$$(3) \qquad {}_ps_{\overline{n}|} = \frac{s_{\overline{np}|}}{s_{\overline{p}|}}, \text{ at rate } \frac{j_{(p)}}{p}.$$

Thus the accumulated amount of an annuity of R a year for n years, with interest converted p times a year at nominal rate $j_{(p)}$ is

$$(4) \qquad K = R\frac{s_{\overline{np}|}}{s_{\overline{p}|}}, \text{ at rate } \frac{j_{(p)}}{p},$$

a convenient formula if $j_{(p)}/p$ is a tabulated rate. If $j_{(p)}/p$ is not a tabulated rate, one must use

$$(5) \qquad K = R\frac{(1+i)^n - 1}{i},$$

and compute $1 + i$ and i from

$$(6) \qquad 1 + i = \left(1 + \frac{j_{(p)}}{p}\right)^p.$$

In a similar manner, if we let ${}_pa_{\overline{n}|}$ be the present value of such an annuity as is being considered, we have

$${}_pa_{\overline{n}|} = a_{\overline{n}|} \text{ at rate } i,$$

so that, by referring to (7), page 50, we have

$${}_pa_{\overline{n}|} = \frac{1 - (1+i)^{-n}}{i}.$$

Introducing the value of $1 + i$ and i in terms of $j_{(p)}$, and proceeding as before, we obtain

$${}_pa_{\overline{n}|} = \frac{a_{\overline{np}|}}{s_{\overline{p}|}} \text{ at rate } \frac{j_{(p)}}{p}.$$

If the annual payment is R, we have for the present value

$$(7) \qquad A = R\frac{a_{\overline{np}|}}{s_{\overline{p}|}}, \text{ at rate } \frac{j_{(p)}}{p},$$

to be used if $j_{(p)}/p$ is tabulated, and

$$A = R\frac{1 - (1+i)^{-n}}{i},$$

where $1 + i$ and i are computed from (6), if $j_{(p)}/p$ is not tabulated.

37. Comment. If a person were asked to write down a formula for ${}_pa_{\overline{n}|}$ "by analogy" with that obtained for ${}_ps_{\overline{n}|}$, he very likely would insert $a_{\overline{p}|}$ in the denominator, instead of the correct quantity $s_{\overline{p}|}$ that is given in (7). The results that have been obtained show the great power of algebraic analysis. On page 62 it was stated that a person should see that the solution of the problem there considered is $s_{\overline{10}|}$ at rate 4.04%, without resorting to series and formal algebraic deduction. But it would be too much to expect a person to see that the result is also $s_{\overline{20}|}/s_{\overline{2}|}$, where both symbols are evaluated for 2%. It is true that at times a person can go through a long formal algebraic derivation where the result arrived at can be obtained readily without such work. An important illustration is the derivation of the formulas for $s_{\overline{n}|}$ and $a_{\overline{n}|}$ given on page 52. But formal algebraic manipulation often leads to very important results that would not be foreseen by unaided thinking. It is impossible to exaggerate the great power of algebra. What has been accomplished in the present discussion is an excellent example. In this case the trick that led to a very significant and unexpected result was the division of the numerator and denominator of (1) by $j_{(p)}/p$, after $1 + i$ and i had been replaced by their values in terms of $j_{(p)}$. The student will probably say that he would not have thought to do that, which is quite true. But by studying such things and noting them well, one can gain ability to make profitable algebraic transformations himself.

Example 1. The sum of \$800 was deposited in a bank at the end of each year, and the account was accumulated at 4% nominal compounded quarterly. How much was in the account at the end of 8 years?

By (4)

$$K = \$800 \frac{s_{\overline{32}|}}{s_{\overline{4}|}} (1\%) = \$7387.27.$$

$$\begin{aligned} \log 800 &= 2.90\,3090 \\ \log s_{\overline{32}|} &= 1.57\,3963 \\ \text{colog } s_{\overline{4}|} &= 9.39\,1431 \\ \hline \log K &= 3.86\,8484 \end{aligned}$$

Example 2. How much must be deposited in a fund that pays 2% nominal converted semiannually, in order to withdraw \$500 at the end of each year for 6 years?

By (7)

$$A = \$500 \frac{a_{\overline{12}|}}{s_{\overline{2}|}} (1\%) = \$2799.77.$$

$$\begin{aligned} \log 500 &= 2.69\,8970 \\ \log a_{\overline{12}|} &= 1.05\,1348 \\ \text{colog } s_{\overline{2}|} &= 9.69\,6804 \\ \hline \log A &= 3.44\,7122 \end{aligned}$$

EXERCISES

1. To how much will \$1250 paid at the end of each year for 15 years accumulate if 3% nominal interest compounded semiannually is allowed?

2. Beginning the first week of January, 1920, and continuing annually until the first week of January, 1928, the sum of \$2000 was deposited in an account paying 4% nominal converted quarterly. What was the value of the account the first week of January, 1930?

3. What is the present value of an annuity of \$750 a year for 12 years on a 2% nominal basis compounded quarterly?

4. How much must be deposited in a bank on April 15, 1912, in order to make 7 yearly withdrawals of \$1200 each, beginning April 15, 1920, if the interest rate is 3% nominal compounded semiannually?

5. State a problem for which the answer would be

$$\$500 \frac{s_{\overline{48}|}}{s_{\overline{4}|}} \ (1\%).$$

6. State a problem for which the answer would be

$$\$750 \frac{a_{\overline{30}|}}{s_{\overline{2}|}} \ (1\tfrac{1}{2}\%).$$

7. State a problem for which the answer would be

$$\$250 \frac{(1.02)^{16} - 1}{(1.02)^{2} - 1} (1.02)^{10}.$$

8. State a problem for which the answer would be

$$\$275 \frac{a_{\overline{24}|}}{s_{\overline{4}|}} v^{16}(\tfrac{1}{2}\%).$$

9. How many yearly payments of \$500 each must be made into a fund in order to accumulate \$12,000, if the fund draws 3% nominal interest converted semiannually?

10. The sum of \$2500 has been deposited in a bank which pays 2% nominal converted quarterly. How many yearly withdrawals of \$500 each can be made, the first withdrawal being made in 6 years, and what final withdrawal less than \$500 can be made?

38. Partial payments. The payments on an annuity may be made more often than interest is converted. For instance, interest may be converted once a year, and the payments on the annuity may be made monthly.

It might seem at first that we have here a new problem that must be attacked directly. A little consideration, however, shows that it can be thrown back upon the basic annuity, and one's understanding of the entire subject is increased by observing carefully how this is accomplished.

Let the annuity be 1 a year paid in p equal installments, each of amount $1/p$. Now although interest is converted only once a year, we can conceive of it as being converted p times a year, at the proper nominal rate. Let $j_{(p)}$ be the nominal rate, which, converted p times a year, is equivalent to the effective rate i. Suppose that the annuity continues over n years (the first payment $1/p$ being made at the end of the first pth fraction of the first year). Let the accumulated amount be represented by $s_{\overline{n}|}^{(p)}$. We then have

$$(8) \qquad s_{\overline{n}|}^{(p)} = \frac{1}{p}\cdot s_{\overline{np}|}\left(\text{at rate } \frac{j_{(p)}}{p}\right).$$

The nominal rate will not likely be one for which the accumulated amount is tabulated in Table V, for when an effective rate is given, it is in all probability one of the customary commensurable rates, and the corresponding nominal rate is incommensurable. We must therefore transform (8) into a more usable form.

From (1), page 45, we have

$$s_{\overline{n}|}^{(p)} = \frac{1}{p}\cdot\frac{\left(1+\frac{j_{(p)}}{p}\right)^{np}-1}{\frac{j_{(p)}}{p}} = \frac{\left(1+\frac{j_{(p)}}{p}\right)^{np}-1}{j_{(p)}}.$$

By (2), page 26,

$$\left(1+\frac{j_{(p)}}{p}\right)^{p} = (1+i),$$

so that

$$\left(1+\frac{j_{(p)}}{p}\right)^{np} = (1+i)^n.$$

We therefore have

$$s_{\overline{n}|}^{(p)} = \frac{(1+i)^n-1}{j_{(p)}}.$$

This result is further transformed by writing

$$(9) \qquad s_{\overline{n}|}^{(p)} = \frac{(1+i)^n-1}{i}\cdot\frac{i}{j_{(p)}}.$$

The first factor on the right in (9) is merely $s_{\overline{n}|}$ at rate i, for yearly payments, so that we have finally the very concise result

$$(10) \qquad s_{\overline{n}|}^{(p)} = s_{\overline{n}|} \cdot \frac{i}{j_{(p)}} \cdot$$

Thus, partial payments are allowed for by multiplying by the factor $i/j_{(p)}$. Table X gives the value of this quantity for different values of i and p. It should be observed that the actual installment payment $1/p$ that appeared in (8) has disappeared in (10).

If the *annual* payment is R, the accumulated amount K is given by the formula

$$(11) \qquad K = Rs_{\overline{n}|}^{(p)} = Rs_{\overline{n}|} \cdot \frac{i}{j_{(p)}} \cdot$$

It is to be noted that even though the annual payment R may be a convenient number (for example, 500, 1000, $\cdots$) there occurs a multiplication of numbers, namely $s_{\overline{n}|}$ and $i/j_{(p)}$, that have a number of significant figures in their decimal portions. Thus the employment of logarithms is indicated. The logarithms of $i/j_{(p)}$ are also given in Table X.

It is again to be noted that a very significant result was obtained by algebraic transformation, by merely inserting the quantity i in the numerator and the denominator of a fraction.

Example 1. Yearly payments of \$2000 are made in equal monthly installments into a fund drawing 3½% interest a year. How much is in the fund at the end of 8 years?

By (11) the amount is

$$K = \$2000 s_{\overline{8}|} \cdot \frac{i}{j_{(12)}} (3\tfrac{1}{2}\%)^* = \$18{,}392.$$

$$\begin{aligned}
&\log 2000 && = 3.30\,1030 && \\
&\log s_{\overline{8}|}(3\tfrac{1}{2}\%) && = 0.95\,6730 && \text{(Table V)} \\
&\log \frac{i}{j_{(12)}} (3\tfrac{1}{2}\%) && = \underline{0.00\,6869} && \text{(Table X)} \\
&\log K && = 4.26\,4629 &&
\end{aligned}$$

The present value of the annuity is treated in a similar manner. For an annuity of 1 per year in p installments a year, over n years, the present value is represented by $a_{\overline{n}|}^{(p)}$. We have

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* Both $s_{\overline{8}|}$ and $i/j_{(12)}$ are to be taken for $i = 3\tfrac{1}{2}\%$.

$$(12) \qquad a_{\overline{n}|}^{(p)} = a_{\overline{n}|} \cdot \frac{i}{j_{(p)}}\,.$$

If the annual payment is R, the present value is

$$(13) \qquad A = Ra_{\overline{n}|}^{(p)} = Ra_{\overline{n}|} \cdot \frac{i}{j_{(p)}}\,.$$

The formula gives the present value one period before the first installment payment is made.

Example 2. How much must be deposited in a bank that pays interest at the rate of 4% effective in order that \$200 can be withdrawn each quarter for 10 years?

The yearly withdrawal being \$800, we have

$$A = \$800a_{\overline{10}|} \cdot \frac{i}{j_{(4)}}\,(4\%) = \$6585.26.$$

$$\begin{aligned}
\log 800 &= 2.90\,3090 \\
\log a_{\overline{10}|}(4\%) &= 0.90\,9069 \\
\log \frac{i}{j_{(4)}}\,(4\%) &= \underline{0.00\,6414} \\
\log A &= 3.81\,8573
\end{aligned}$$

39. The more general case. It is possible to have a nominal rate and installment payments, where the intervals in question are not the same. The most general case would have considerable complexity, but the instances most likely to occur are such as, for example, where money is converted semiannually and installments on the annuity are paid monthly. We consider then only cases where payments on the annuity are more frequent than conversion of interest, and where there is an integral number of payment periods in a conversion period.

It is possible to set up appropriate formulas, but formulas (11) and (13) can be used by proper interpretations, and it is better to proceed in such a manner, for then certain problems are seen to be essentially related instead of appearing as different. In order to use (11) and (13) it is merely necessary to observe that R is the payment per conversion period, that n is the number of conversion periods, that p is the number of payments per conversion period, that i is the rate per conversion period, and that $j_{(p)}$ is the corresponding nominal rate.

Example. Deposits of \$100 were made at the end of each month in a bank that paid 4% nominal converted semiannually. How much was in the account at the end of 7 years?

The payment per conversion period is \$600, and there are 14 such periods. The rate per conversion period is 2%, and there are 6 installments on the annuity per period. (The accumulated amount will, in fact, be the same as it would be if \$100 were paid bimonthly, over a period of 14 years, the bank paying 2% per year.) Hence

$$K = \$600 s_{\overline{14}|} \cdot \frac{i}{j_{(6)}}(2\%) = \$9663.92.$$

$$\begin{aligned} \log 600 &= 2.77\,8151 \\ \log s_{\overline{14}|}(2\%) &= 1.20\,3412 \\ \log \frac{i}{j_{(6)}}\,(2\%) &= \underline{0.00\,3590} \\ &\ 3.98\,5153 \end{aligned}$$

The solutions of the problems considered in this and the preceding section are notably simple. In case one has to do with a certain interest rate and a certain number of payments per year, with a large number of accounts to handle, a special table can be readily constructed that will greatly shorten the work to be done. Thus suppose that a savings bank, paying 2.5% annually, wishes to encourage monthly deposits. A table of $s_{\overline{n}|} \cdot i/j_{(12)}$ (for $i = 2.5\%$) can be constructed by multiplying the constant $i/j_{(12)}$ by $s_{\overline{n}|}$ for different values of n. The amount of any depositor's account at the end of any year can then be ascertained quickly by multiplying the yearly deposit by a number taken from the table.

It is to be noted that in the solution that has been given, interest for portions of a period are taken according to the compound interest formula. Interest for fractions of a period can be taken by simple interest, and a solution of annuity problems obtained. See Exercise 28 that follows. The solution by compound interest is, however, more customarily made by persons with a knowledge of investment theory.

EXERCISES

1. If \$50 is deposited in a bank at the end of each month for 10 years, how much will be in the account at the end of the time if the bank pays 1½% effective?

2. The sum of \$35 was deposited at the end of each quarter in a bank that paid 3% effective. How much was in the account at the end of 8 years?

3. Every 6 months beginning July 1, 1920, and continuing until July 1, 1928, inclusive, the sum of $350 was deposited in a bank paying 3½% effective. How much was in the account on July 1, 1934?

4. Beginning at the end of February, 1920, the sum of $325 was paid every 2 months into a fund that was being accumulated at 4% effective. How much was in the fund after the payment on December 31, 1929?

5. What is the purchase price on a 2% effective basis of an annuity that pays $200 quarterly and runs for 6 years, if the first payment is received in 3 months? in 15 months? in 12 months?

6. On July 1, 1924, a house was bought for $2000 cash and $40 a month, beginning August 1, 1924, and continuing for 5 years. What was the equivalent cash price on a 6% effective basis?

7. A man paid $10,000 for an annuity with monthly payments to continue through 10 years, the first payment to be received in a month. What was the monthly payment on a 3% effective basis?

8. State problems for which the respective answers would be

$$K = \$1000 \cdot s_{\overline{6}|} \cdot \frac{i}{j_{(4)}} (4\%),^{*} \qquad K = \$250 \cdot s_{\overline{24}|}(1\%),$$

and calculate the two amounts.

9. State problems for which the answers would be respectively

$$A = \$6000 \cdot a_{\overline{8}|} \cdot \frac{i}{j_{(2)}} (5\%), \qquad A = \$3000 \cdot a_{\overline{16}|}(2\tfrac{1}{2}\%),$$

and calculate the two amounts.

10. State a problem for which the answer would be

$$\$240 \cdot s_{\overline{8}|} \frac{i}{j_{(12)}} (1.02)^{5/4}(2\%),$$

and calculate the amount.

11. What is the purchase price on a 4% effective basis of an annuity that pays $1000 semiannually and runs for 10 years, if the first payment is received in 1 year?

12. State a problem for which the answer would be

$$\$600 \cdot a_{\overline{4}|} \cdot \frac{i}{j_{(12)}} v^{59/12}(3\%).$$

13. If $200 is saved every quarter and is deposited in an account paying 3½% effective, how long will it require to accumulate $10,000 and what will be the amount of the last deposit?

Solution: We have

$$800 \cdot s_{\overline{n}|} \cdot \frac{i}{j_{(4)}} (3\tfrac{1}{2}\%) = 10{,}000,$$

$$s_{\overline{n}|} = \frac{12.5}{\dfrac{i}{j_{(4)}}} = \frac{12.5}{1.0130} = 12.34.$$

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[*] In all cases a rate of interest applies to all preceding symbols that do not show the rate.

From the table it is seen that n is between 10 and 11. At the end of 10 years there is in the account

$$\$800 \cdot s_{\overline{10|}} \cdot \frac{i}{j_{(4)}} (3\tfrac{1}{2}\%) = \$9507.42.$$

The amount in the account before the 41st payment is

$$\$9507.42(1.035)^{1/4} = \$9589.53,$$

so that the amount after the payment is $9789.53. The amount before the 42nd payment is

$$\$9789.53(1.035)^{1/4} = \$9874.10.$$

Therefore a 42nd payment of amount $125.90 is necessary. The student should verify all calculations.

14. If $750 is saved every 6 months and deposited in a fund paying 2% effective, how long will be required to accumulate $20,000, and what will the final payment be?

15. The sum of $18,000 is deposited in a fund paying 3½% effective, and $250 is withdrawn quarterly, the first withdrawal being made in 3 months. How many withdrawals can be made and what will the final withdrawal less than $250 be?

16. Equal deposits were made at the end of each quarter for 10 years in a bank paying 2½%, the last deposit being on December 31, 1922. Monthly withdrawals of $50 each were started January 31, 1928, and were to continue for 5 years, when the account was to be exhausted. What was the quarterly deposit?

17. Monthly deposits of $40 were made over a period of 8 years in a bank paying 3½%. What quarterly withdrawal over a period of 4 years can be made, the first withdrawal being (a) 3 months after the last deposit, (b) 4½ years after the last deposit?

18. The sum of $100 was deposited at the end of each month for 8 years in a bank paying 4% effective. What was the value of the account 2 years after the last deposit, no withdrawals having been made?

19. How much must be deposited January 1, 1936, in a bank paying 2½% effective so that monthly withdrawals of $500 can be made for 6 years, the first withdrawal being February 1, 1940, and the last withdrawal exhausting the account?

20. On January 1, 1920, $15,000, was deposited in a bank paying 3½% effective. Monthly withdrawals of $50 each were made over a period of 4 years, the first withdrawal being made February 1, 1928. What balance was in the account after the last withdrawal?

21. Monthly deposits of $25 were made for 10 years in a bank paying 3%, the first deposit being made January 31, 1910. On December 31, 1922, $500 was withdrawn, and on December 31, 1924, $200 was withdrawn. The balance of the account was withdrawn December 31, 1929. What was the balance?

22. Find the amount of an annuity of \$2500 a year, paid in quarterly installments, if the interest rate is 3% nominal converted semiannually, the annuity running for 6 years.

23. Find the amount of an annuity of \$600 a year, paid in monthly installments, the interest rate being 2% nominal converted quarterly, the annuity running for 10 years.

24. What is the present value of an annuity of \$500 a month (the first payment made in one month) running for 8 years, if the interest rate is 4% nominal converted semiannually?

25. If monthly deposits of \$50 for 5 years accumulate to \$3200, what is the effective rate?

SOLUTION. We first neglect the factor $i/j_{(12)}$ and assume yearly deposits of \$600. We then have

$$600 s_{\overline{5}|} = 3200, \qquad s_{\overline{5}|} = 5.333.$$

The rate is seen to be close to 3%. We then compute

$$\begin{aligned} 600 s_{\overline{5}|} \frac{i}{j_{(12)}} \,(3\%) &= 3229.04, \\ 600 s_{\overline{5}|} \frac{i}{j_{(12)}} \,(2\tfrac{1}{2}\%) &= 3189.77. \end{aligned}$$

The required rate can now be found by interpolation.

26. If monthly payments of \$25 accumulate in 10 years to \$3600, find the effective interest rate.

27. The present value of an annuity of \$100 a month for 10 years is \$10,000; find the effective rate.

28. If a payment of 1/12 is made at the end of each month and if simple interest is used, show by a procedure similar to that in section 25 that the accumulated amount at the end of the year is

$$1 + \frac{11}{24} i.$$

Using this result, show that we would have for the accumulated amount of an annuity of R a year, at the end of n years,

$$R s_{\overline{n}|} \left(1 + \frac{11}{24} i\right),$$

if monthly payments are made.

Compare the factor $(1 + 11i/24)$ with $i/j_{(12)}$ for different values of i.

29. Find the accumulated amount in Example 1, page 68, by Exercise 28.

30. By using the binomial theorem show that

$$\frac{i}{j_{(12)}} = 1 + \frac{11}{24} i - \frac{143}{1728} i^2 + \frac{143}{3456} i^3 + \cdots,$$

and thus explain the numerical differences obtained in Exercise 28.

CHAPTER VI

SINKING FUNDS AND AMORTIZATION

40. The annuity that will amount to 1. There are many cases in which the quantity to be found is the periodic payment R of an annuity. So important are such problems that this entire chapter is devoted to them.

If in (4), page 46, we put $K = 1$, we have

$$R_1 = \frac{1}{s_{\overline{n}|}}. \tag{1}$$

Thus $1/s_{\overline{n}|}$ *is the annuity that in n payments will accumulate to* 1. We have used the subscript 1, writing R_1, merely to prevent confusion with the next equation to be written, which will be the one generally used. The only purpose in writing the equation above is to call attention to the meaning of the reciprocal $1/s_{\overline{n}|}$.

In general, if K, n, and i are given, the required annuity payment is

$$R = \frac{K}{s_{\overline{n}|}} = K \cdot \frac{1}{s_{\overline{n}|}}, \tag{2}$$

where $s_{\overline{n}|}$ is evaluated for the rate i.

41. The annuity that 1 will buy. If in (9), page 51, we put $A = 1$, we see that the periodic payment of the annuity is $1/a_{\overline{n}|}$. *Thus* $1/a_{\overline{n}|}$ *is the annuity that* 1 *will buy.*

In general, if A, n, and i are given we have

$$R = \frac{A}{a_{\overline{n}|}} = A \cdot \frac{1}{a_{\overline{n}|}}, \tag{3}$$

where $a_{\overline{n}|}$ is at the rate i. This formula gives the annual income that can be purchased for the single payment A, the first payment from the annuity being received in 1 year, and there being n payments in all.

42. Relation between $1/s_{\overline{n}|}$ and $1/a_{\overline{n}|}$. There is a simple relation between the annuity that will amount to 1 and the annuity that 1 will buy. It can be obtained by using the formulas for $s_{\overline{n}|}$ and $a_{\overline{n}|}$, but such a derivation will be left to the exercises. We derive the relation here by reasoning on the meaning of the quantities. The student is urged to master the thinking involved, for this will aid his grasp of the subject, and when fully understood the relation will prove to be a most interesting one.

Suppose that a person uses the capital 1 to purchase an annuity for n years, the annuity being computed at rate i. He will then receive n payments of amount $1/a_{\overline{n}|}$. If he spends all this income he will be gradually using up his initial capital. He is, in fact, entitled to spend yearly only the interest i, for this is all that is actually earned by the original amount 1. Suppose that the person puts the balance of each payment of the annuity, namely

$$\frac{1}{a_{\overline{n}|}} - i,$$

into a bank that pays interest at rate i. When he has received the last payment of the annuity his entire account with the annuity company will have been exhausted. But since he has been spending only the actual earning of his original capital and has not been encroaching upon the capital, it is clear that his account with the bank, in which he has been regularly depositing the amount noted above, must be exactly 1. But we know that the regular payment required to amount to 1 upon n payments is $1/s_{\overline{n}|}$.

Hence it follows that

$$\frac{1}{s_{\overline{n}|}} = \frac{1}{a_{\overline{n}|}} - i. \tag{4}$$

The relation (12), page 53, and that just obtained give two equations between the fundamental quantities $s_{\overline{n}|}$ and $a_{\overline{n}|}$, from which the explicit formulas given in (3), page 45, and (7), page 50, can be easily obtained by solving the equations simultaneously. The student should carry through the work.

43. Tables. The given amount K in (2) or the given amount A in (3) is often a convenient number. In such instances computation will be facilitated by having a table of values for $1/s_{\overline{n}|}$ or $1/a_{\overline{n}|}$, so that R can be found by a simple multiplication instead of by a more tedious division.

From equation (4) it follows that we need a table for only one of the quantities $1/s_{\overline{n}|}$ and $1/a_{\overline{n}|}$. Table VII is a table for $1/a_{\overline{n}|}$. When $1/s_{\overline{n}|}$ is desired, it is merely necessary to subtract i from $1/a_{\overline{n}|}$.

Example 1. How much must be deposited annually in a fund that pays 5% in order to have \$100,000 in 25 years?

We have

$$R = \$100{,}000 \frac{1}{s_{\overline{25}|}} (5\%) = \$100{,}000(.020\,9525)$$
$$= \$2095.25.$$

(From the table, $1/a_{\overline{25}|} = .070\,9525$. Hence $1/s_{\overline{25}|} = .020\,9525$.)

Example 2. What annuity running for 12 years can be purchased for \$10,000 if the fund draws 2%, the first payment being made in a year?

We have

$$R = \$10{,}000 \frac{1}{a_{\overline{12}|}} (2\%) = \$10{,}000(.094\,5596)$$
$$= \$945.60.$$

In case K in (2) or A in (3) is not a number by which multiplication can be made quickly, the table for $1/s_{\overline{n}|}$ and $1/a_{\overline{n}|}$ is of no special advantage. It may be better to use logarithms, taking log $s_{\overline{n}|}$ from Table V and log $a_{\overline{n}|}$ from Table VI, as the case may be.

Example 3. By the terms of a will the sum of \$240,360 was left to a college and deposited in a bank paying 4%. The entire amount, principal and interest, was to go to the school in 30 equal yearly payments. What was the yearly payment?

We have

$$R = \frac{\$240{,}360}{a_{\overline{30}|}(4\%)} = \$13{,}900.$$

$$\begin{aligned} \log 240{,}360 &= 5.38\,0862 \\ \log a_{\overline{30}|}(4\%) &= 1.23\,7846 \qquad \text{(Table VI)} \\ \hline \log R &= 4.14\,3016 \end{aligned}$$

44. Sinking funds. When a corporation or some unit of government borrows money, it may contract to repay the entire loan at a certain future date. If this is the case, in addition to paying the regular interest charge the debtor should establish a *sinking fund* for the repayment of the principal when it falls due. Into this fund there are regularly paid amounts which, with the interest earned by the fund, will be sufficient to discharge the debt when it becomes due.

The original debt is usually represented by bonds. The interest that is paid on the bonds is usually greater than the interest received on the sinking fund, since the latter is normally held in a bank. If the company issuing the bonds desires to retire some of the bonds before the date specified on them, and thereby to avoid interest charge, it must purchase bonds in the open market. For in the case we are considering the holder of a bond need not surrender it until the maturity date. The price the company will have to pay for bonds in the open market will depend upon general economic and investment conditions, and whether it is advantageous to buy up bonds for retirement must in general be settled by the facts in any special case.

Example. A sinking fund was created to pay a debt of $10,000 due in 6 years, the sinking fund earning 4%. What was the yearly payment into the sinking fund? That was the yearly charge if the debt drew 6%?

The amount to be paid each year into the sinking fund was

$$R = \$10{,}000\,\frac{1}{s_{\overline{6}|}(4\%)} = \$10{,}000\cdot(.150\;7619) = \$1507.62.$$

The yearly interest being $600, the total yearly charge was $2107.62.

A schedule showing the operation of the sinking fund follows.

Year	In Fund at Beginning of Year	Interest Earned	Sum	Yearly Payment	In Fund at End of Year
1	0	0	0	$1507.62	$1507.62
2	$1507.62	$ 60.30	$1567.92	1507.62	3075.54
3	3075.54	123.02	3198.56	1507.62	4706.18
4	4706.18	188.25	4894.43	1507.62	6402.05
5	6402.05	256.08	6658.13	1507.62	8165.75
6	8165.75	326.63	8492.38	1507.62	10000.00

45. Amount in sinking fund at any time. It is not necessary to have a complete schedule in order to tell how much is in the sinking fund at any time. The amount in question is merely the accumulated amount of the payments already made.

Example. A debt of \$20,000 due in 20 years was being provided for by a sinking fund drawing 4½% interest. How much was in the fund after the 14th payment?

The amount was

$$\$20{,}000 \cdot \frac{1}{s_{\overline{20}|}(4\tfrac{1}{2}\%)} \cdot s_{\overline{14}|}(4\tfrac{1}{2}\%) = \$12{,}070.$$

$$\begin{aligned} \log 20{,}000 &= 4.30\ 1030 \\ \text{colog } s_{\overline{20}|}(4\tfrac{1}{2}\%) &= 8.50\ 3466 \\ \log s_{\overline{14}|}(4\tfrac{1}{2}\%) &= 1.27\ 7199 \\ \hline & 4.08\ 1695 \end{aligned}$$

46. Partial payments into sinking fund. The amount earned by the sinking fund will be increased somewhat if partial payments are made into it, and there may be cases where this can be done.

Suppose that it is desired to make p payments a year. From (11), page 68, we have upon solving for R,

$$R = K \cdot \frac{1}{s_{\overline{n}|}} \cdot \frac{j_{(p)}}{i}.$$

It is to be remembered that R here is the annual payment, so that the *periodic* payment is

$$\frac{R}{p} = \frac{K}{p} \cdot \frac{1}{s_{\overline{n}|}} \cdot \frac{j_{(p)}}{i}. \tag{5}$$

EXERCISES

A

1. Prove equation (4) by using the expressions for $s_{\overline{n}|}$ and $a_{\overline{n}|}$ given in (3), page 45, and (7), page 50, respectively.

2. A sinking fund with yearly payments was established to pay a debt of \$12,000 due in 5 years. What was the yearly payment if the fund paid 3½%? Construct a schedule. What was the yearly charge if 6½% interest was paid on the debt?

3. A company has borrowed \$50,000 for which it sold bonds due in 15 years. It desires to construct a sinking fund with a bank paying 2% nominal converted semiannually, making payments into the fund every 6 months. What will the periodic payment be, and how much will be in the fund at the end of 9 years?

4. In the last exercise, what would be the payment into the fund if it were made annually, the fund paying 2% nominal converted semiannually? See section 36, Chapter V.

5. It is desired to pay a debt of \$8000 due in 7 years by monthly payments into a sinking fund that pays $3\frac{1}{2}\%$ effective. What will be the monthly payment?

6. A company borrowed \$200,000, issuing bonds due in 20 years, and set up a sinking fund into which it made quarterly payments. What was the amount of each payment if the fund paid 4% effective?

7. A city constructed a school building costing \$250,000 and sold 5% bonds due in 30 years. It used a sinking fund earning 3%. What was the increase of city taxation each year due to the building?

8. A corporation sold 20-year bonds to the value of \$1,000,000, and established a sinking fund that earned 4%, into which yearly payments were to be made. The first 15 payments were made, then because of adverse earnings payments into the fund were reduced to \$10,000 a year Show that when the bonds were due the company had to refinance to the extent of

$$\$1{,}000{,}000 - \frac{\$1{,}000{,}000}{s_{\overline{20}|}}(s_{\overline{20}|} - s_{\overline{5}|})(4\%) - \$10{,}000 s_{\overline{5}|}(4\%)$$
$$= \$10{,}000 s_{\overline{5}|}\left(\frac{100}{s_{\overline{20}|}} - 1\right)(4\%).$$

Compute the amount.

9. Suppose that \$12,000 is to be paid in 15 years, and that for the first 5 years \$1000 is paid into a sinking fund earning 4%. What equal annual payments during the last 10 years will complete the fund?

10. A company had borrowed \$200,000 by selling 25-year bonds. It set up a sinking fund into which yearly payments were made, the fund earning 4%. The first 12 payments were made, then due to adverse conditions no payments into the fund were made for 3 years. What would be the new annual payments to complete the fund, if the first new payment was made one year after the third payment missed?

11. State problems for which the answers would be

$$\frac{\$40{,}000}{s_{\overline{25}|}} s_{\overline{16}|}(1.04)^5(4\%), \qquad \$25{,}000\left(1 - \frac{s_{\overline{10}|}}{s_{\overline{30}|}}\right)(5\%).$$

B

1. Exercise 1, list A.

2. A sinking fund with yearly payments was to be established to pay a debt of \$22,000 due in 6 years. What was the yearly payment if the fund paid 4%? Construct a schedule. What was the yearly charge if 6% interest was paid on the debt?

3. A company has borrowed \$75,000 for which it sold bonds due in 20 years. It desires to construct a sinking fund with a bank paying 3% nominal converted semiannually, making payments into the fund every 6 months. What will the periodic payment be, and how much will be in the fund at the end of 12 years?

4. In the last exercise, what would be the payment into the fund if it were made annually, the fund paying 3% nominal converted semiannually? See section 36, Chapter V.

5. It is desired to pay a debt of $12,000 due in 9 years by monthly payments into a sinking fund that pays 4% effective. What will the monthly payment be?

6. A company borrowed $100,000, issuing bonds due in 25 years, and set up a sinking fund into which it made quarterly payments. What was the amount of each payment if the fund paid 4% effective?

7. A city constructed a school building costing $325,000 and sold 5½% bonds due in 20 years. It used a sinking fund earning 4%. What was the increase of taxation each year due to the building?

8. A corporation sold 25-year bonds to the value of $2,000,000 and established a sinking fund that earned 4½% into which yearly payments were to be made. The first 15 payments were made, then because of adverse earnings payments into the fund were reduced to $25,000 a year. Show that when the bonds were due the company had to refinance to the extent of

$$\$25{,}000 s_{\overline{10}|} \left(\frac{80}{s_{\overline{25}|}} - 1\right) (4\tfrac{1}{2}\%).$$

(See Exercise 8, list A.) Compute the amount.

9. Suppose that $18,000 is to be paid in 12 years, and that for the first 4 years $2000 is put into a sinking fund earning 2½%. What equal annual payments during the remaining years will complete the fund?

10. A company had borrowed $150,000 by selling 20-year bonds. It set up a sinking fund into which yearly payments were made, the fund earning 3½%. The first 10 payments were made, then because of adverse conditions no payments were made into the fund for 4 years. What would be the new annual payments to complete the fund, if the first new payment were made 1 year after the 4th payment missed?

47. Amortization. In the sinking fund method of paying a debt, the interest on the entire original debt is regularly paid, and at the same time a fund is accumulated that will finally pay off the principal. There is an evident objection to this method, already noted. A larger rate of interest is usually paid on the debt than is received on the sinking fund.

An alternative procedure presents itself. More money may be paid each year to the person making the loan than is required to meet the interest charge, the excess serving to reduce the outstanding principal. If the regular payments are the same, as is likely to be the case, the interest requirement steadily diminishes; more of the payment goes on the principal, which will finally be extinguished. Such a method of payment is called *amortization.*

It is not always possible to arrange to discharge a debt by amortization. The plan must meet with the approval of the person making the loan. If the money has come from selling bonds, it may be necessary to redeem the called bonds at a figure above par, for the purchaser of a bond may prefer to know definitely when it is to be redeemed, so that he will not quite suddenly be faced with the problem of reinvestment. A premium when the bond is called may therefore be a necessary inducement in order to sell it. On the other hand, a person who makes an ordinary loan and allows it to be paid by amortization may squander his principal, for every payment he receives is part interest—that is, actual earning—and part principal. Unless he is careful to add to his capital fund the portion that is return of principal, he will evidently be encroaching upon his permanent resources.

Let there be a debt A drawing interest at rate i, and let it be desired to pay off the debt, principal and interest, in n annual payments each of amount R. The present value of the future payments must be A, so that

$$R \cdot a_{\overline{n}|} = A,$$

or

$$R = \frac{A}{a_{\overline{n}|}} = A \cdot \frac{1}{a_{\overline{n}|}} \cdot \tag{6}$$

In this scheme of reasoning there is nothing that shows that a part of each payment R goes toward interest on the entire outstanding obligation, and the remainder toward lessening the principal. In fact, each payment R actually is thought of as paying a certain part of the original principal and the accumulated interest on that part, without paying interest on the remainder of the debt. That is, the first payment pays a little of the principal and the interest on it for one year. The second payment pays a second part of the original principal and the interest thereon compounded for two years, and so on. But this results in having the debt paid with interest, and it evidently must be possible to conceive of part of each payment as meeting the interest on the entire debt then outstanding, the remainder applying on the principal. A schedule in any case shows this clearly.

If A in (6) is a convenient number, it is simplest to multiply by the reciprocal $1/a_{\overline{n}|}$, which is given in Table VII. If this is not the case, then logarithms may be used, log $a_{\overline{n}|}$ being taken from Table VI.

Example 1. What regular annual payment is required to amortize in 5 payments a debt of \$5000 bearing interest at 6%? Construct a schedule.

We have for the annual payment

$$R = \$5000 \frac{1}{a_{\overline{5}|}(6\%)} = \$5000(.237\ 3964) = \$1186.98.$$

The schedule follows:

YEAR	DEBT AT BEGINNING OF YEAR	INTEREST CHARGE	PAYMENT	APPLIED TO PRINCIPAL	DEBT AT END OF YEAR
1	\$5000.00	\$300.00	\$1186.98	\$886.98	\$4113.02
2	4113.02	246.78	1186.98	940.20	3172.82
3	3172.82	190.37	1186.98	996.61	2176.21
4	2176.21	130.57	1186.98	1056.41	1119.80
5	1119.80	67.19	1186.99	1119.80	0.00
				5000.00	

The person making the loan should have a schedule in order to tell how much of each payment should go into his capital account.

Example 2. What regular semiannual payment will in 10 years pay principal and interest on a debt of \$6375, interest being at 7% nominal, converted semiannually?

We have for the semiannual payment

$$R = \frac{\$6375}{a_{\overline{20}|}(3\tfrac{1}{2}\%)} = \$448.55.$$

$$\begin{aligned} \log 6375 &= 3.80\ 4480 \\ \log a_{\overline{20}|}(3\tfrac{1}{2}\%) &= 1.15\ 2668 \\ \hline \log R &= 2.65\ 1812 \end{aligned}$$

48. Principal outstanding at any time. The principal outstanding at any time is merely the present value of the future payments, and can be calculated without having a complete schedule.

Example. A debt of \$10,000 with interest at $5\tfrac{1}{2}\%$ is being amortized in 15 payments. The 9th payment has just been made. What is the outstanding principal?

There being 6 remaining payments, the required amount is

$$\frac{\$10{,}000}{a_{\overline{15}|}(5\tfrac{1}{2}\%)}\cdot a_{\overline{6}|}(5\tfrac{1}{2}\%) = \$4976.83.$$

$$\begin{aligned}
\log 10{,}000 &= 4.00\ 0000\\
\operatorname{colog} a_{\overline{15}|}(5\tfrac{1}{2}\%) &= 8.99\ 8371\\
\log a_{\overline{6}|}(5\tfrac{1}{2}\%) &= 0.69\ 8582\\
\hline
&\ 3.69\ 6953
\end{aligned}$$

After the 9th payment has been received, the person making the loan should have returned in all \$10,000 − \$4976.83 = \$5023.17 to his capital account.

49. Partial payments in amortization. There may be occasions where it is desired to make payments more frequently than once a year. In this way the debtor has the opportunity to pay his debt as he saves for it, and there will be a decrease in the total yearly payment.

Let there be p payments a year. From (13), page 69, we have upon solving for R,

$$R = A\cdot\frac{1}{a_{\overline{n}|}}\frac{j_{(p)}}{i},$$

as the annual payment.

The *periodic* payment is therefore

$$\frac{R}{p} = \frac{A}{p}\cdot\frac{1}{a_{\overline{n}|}}\cdot\frac{j_{(p)}}{i}. \tag{7}$$

Considerable computation may be necessary to construct a schedule in the case of partial payments, for interest for a fraction of a year is to be computed by the compound interest formula.

EXERCISES

A

1. A debt of \$9000 with interest at 6% is to be paid, principal and interest, in 8 equal annual payments. Find the amount of each payment, and construct a schedule.

2. What regular quarterly payment will in 12 years amortize a debt of \$12,000 with interest at 5% nominal, compounded quarterly?

3. A debt of \$12,450 with interest at 4½% effective is to be amortized by equal quarterly payments over 7 years. Find the amount of each payment.

4. The cash price of a house is \$15,000. It has been sold for \$3000 down and equal monthly payments extending over 12 years, with 6% effective interest on all principal unpaid. What is the monthly payment?

5. A debt of \$20,000 with interest at 5% was being paid, principal and interest, in 15 annual installments. The debtor had the right to pay the entire debt at the time of any annual payment. He wished to avail himself of the privilege at the time the 12th payment was due. How much should he have paid?

6. The paving assessment on a piece of property was \$1235.14. It was being paid in 8 equal annual installments with interest at 5½%. When the 5th payment was due, the property owner wished to settle the obligation in full. How much did he owe the city?

7. Certain trust funds to the amount of \$20,000 had been lent at 4% nominal interest converted semiannually. The loan was being amortized by payments every 6 months, extending over 10 years. Just after the 15th payment had been received, what was the *book value* * of the remaining payments, and how much should have been returned to capital?

8. A man offered a house for \$10,000, but accepted \$2000 cash and 6 equal yearly payments to include interest at 7%. Make a schedule to show how much he should take out of each payment in order to maintain his \$10,000 asset.

9. Compare the yearly payment necessary to amortize in 12 years a debt of \$25,000 bearing interest at 6%, with the yearly payment necessary to cover interest and pay the principal in 12 years if a sinking fund earning 4½% is used.

10. A debt of \$15,000 is to be paid in 12 equal annual payments. Interest for the first 6 years is to be at 5%, and for the last 6 years at 6%. Show that the equation of value for R, the yearly payment, is

$$R\left[a_{\overline{6}|}(5\%) + a_{\overline{6}|}(6\%)\, v^{6}(5\%)\right] = \$15{,}000.$$

Find R.

11. A debt of \$25,000 is to be amortized in 15 equal yearly payments. Interest for the first 5 years is to be at 5%; for the next 5 years it is to be at 6%, and for the last 5 years it is to be at 7%. Find the yearly payment.

12. After the payment of a cash amount on a piece of property there remained a balance of \$8000, which was to be paid in 10 years with interest at 5½%. The payments for the last 5 years were to be double those for the first 5 years. Find the two different annual payments.

13. A debt of \$11,000 with interest at 5½% was being paid, principal and interest, in 20 annual payments. The first 8 payments were made, then for 3 years only the interest was paid. What was the interest?

14. In Exercise 13, suppose that payments were completely defaulted for the 3 years in question, and that the succeeding year all defaulted payments were paid with accumulated interest, as well as the payment then due. What was the payment made?

15. A debt of \$500,000 with interest at 5% was being paid, principal and interest, in 20 equal yearly payments. The first 10 payments were made.

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* The *book value* would be the amount of the original principal still unpaid.

For the next 4 years only interest was paid. A new plan was then drawn up, providing for the remaining debt to be amortized in 10 annual payments, with the interest rate reduced to $4\frac{1}{2}\%$, the first of the new payments being made one year after the 4th partial payment. Find the new annual payment.

16. A debt of \$1,000,000 with interest at 4% was being amortized over a 30-year period. The first 15 payments were made, the next 4 were defaulted. A new agreement was then made to pay the remaining obligation in 20 equal annual payments with interest at $3\frac{1}{2}\%$, the first payment being made 1 year after the 4th defaulted payment. Show that the amount of the new yearly payments was

$$\frac{\$1{,}000{,}000}{a_{\overline{30}|}(4\%)}\cdot[s_{\overline{4}|}(4\%)+a_{\overline{11}|}(4\%)]\cdot\frac{1}{a_{\overline{20}|}(3\frac{1}{2}\%)}.$$

Compute the amount.

17. Interpret the equation of value

$$A\cdot s_{\overline{11}|}(4\%) = \$1000\cdot a_{\overline{5}|}(6\%).$$

18. Interpret the equation of value

$$A\cdot s_{\overline{r}|}(\text{rate } i) = P\cdot a_{\overline{q}|}(\text{rate } i')$$

if (a) r, q, and P are given; (b) r, q, and A are given; (c) r, A, and P are given.

19. Interpret the equation of value

$$A\cdot s_{\overline{11}|} = \$2000(a_{\overline{20}|} - a_{\overline{16}|}). \quad (4\%)$$

20. Interpret the equation of value

$$A(s_{\overline{15}|} - s_{\overline{4}|}) = \$1500\cdot a_{\overline{5}|}. \quad (5\%)$$

B

1. A debt of \$12,000 with interest at 7% is to be paid, principal and interest, in 6 equal annual payments. Find the annual payment, and construct a schedule.

2. What regularly quarterly payment will in 10 years amortize a debt of \$15,000 with interest at 7% nominal compounded quarterly?

3. A debt of \$18,575 with interest at 6% effective is to be amortized by equal quarterly payments over 10 years. Find the amount of each payment

4. The cash price of a house was \$12,000. It has been sold for \$2000 down and equal monthly payments extending over 8 years, with 3% effective interest on all principal unpaid. What is the monthly payment?

5. A debt of \$20,000 with interest at 4% was being paid, principal and interest, in 12 annual installments. The debtor had the right to pay the entire debt at the time of any annual payment. He wished to avail himself of the privilege at the time of the 8th payment. How much should he have paid?

6. The paving assessment on a piece of property was \$1546.50. It was being paid in 10 equal annual installments with interest at 6%. When the 7th installment was due the property owner wished to settle the obligation in full. How much did he owe the city?

7. Exercise 7, list A.

8. A man offered a house for $19,000, but accepted $4000 cash and 6 equal yearly payments to include interest at 6%. Make a schedule to show how much he should take out of each payment in order to maintain his $19,000 asset.

9. Compare the yearly payment necessary to amortize in 10 years a debt of $20,000 bearing interest at 5%, with the yearly payment necessary to cover interest and pay the principal in 10 years if a sinking fund earning 2% is used.

10. Exercise 10, list A.

11. A debt of $25,000 is to be amortized in 12 equal yearly payments. Interest for the first 4 years is to be at 5%; for the next 4 years it is to be at 5½%, and for the last 4 years at 6%. Find the yearly payment.

12. After the payment of a cash amount on a piece of property there remained a balance of $12,000, which was to be paid in 10 years with interest at 7%. The payments for the last 5 years were to be double those for the first 5 years. Find the two different annual payments.

13. A debt of $22,000 with interest at 6% was being paid, principal and interest, in 15 annual payments. The first 6 payments were made, then for 4 years only the interest was paid. What was the interest?

14. In Exercise 13, suppose that payments were completely defaulted for the 4 years in question, and that the succeeding year all defaulted payments were paid with accumulated interest as well as the payment then due. What was the payment made?

15. A debt of $750,000 with interest at 2½% was being paid, principal and interest in 25 equal yearly payments. The first 15 payments were made. For the next 3 years only interest was paid. A new plan was then drawn up, providing for the remaining debt to be amortized in 10 annual payments, with the interest rate reduced to 5%, the first of the new payments being made one year after the 3rd partial payment. Find the new annual payment.

16. to **20.** Exercise, 16-20, list A.

50. Amortization of a bonded debt. If money is raised by the sale of bonds, it is sometimes provided that the bonds are subject to call on any interest date. In such cases it may be necessary either to make the interest rate high, or to provide that a premium be paid when the bonds are called, in order to overcome the reluctance, already commented upon, that a purchaser might have in buying a bond whose redemption date was not definitely set. For instance, a $100 bond might be retired at $105. Then, if it is called in a year or two, the purchaser will have made in a short time a distinct increase in capital. The outlay each year for retiring bonds must be an integral multiple of the redemption value of a single bond, and consequently it is not possible to pay every year the same amount toward interest and retiring the issue. It is possible, however, to keep the yearly payment quite nearly constant.

Example. The sum of \$50,000 has been raised by selling 50 bonds each of \$1000 denomination. The bonds pay 6% interest and are to be retired by call over a period of 10 years. Construct a schedule for the retirement of the bonds.

The amount required to amortize a debt of \$50,000 with interest at 6% in 10 equal yearly payments is

$$R = \$50{,}000\,\frac{1}{a_{\overline{10|}}(6\%)} = \$6793.40.$$

Each year the number of bonds selected for retirement should be so chosen that the total interest charge and the outlay for retiring bonds together amount to as near \$6793 as possible. Thus the first year the retirement of 4 bonds is clearly indicated, for with an interest charge of \$3000 the total outlay will be \$7000. The schedule is continued in a self-explanatory manner.

Year	Debt at Beginning of Year	Interest	Number of Bonds Retired	Principal Repaid	Total Payment	Debt at End of Year
1	\$50,000	\$3000	4	\$4000	\$7000	\$46,000
2	46,000	2760	4	4000	6760	42,000
3	42,000	2520	4	4000	6520	38,000
4	38,000	2280	5	5000	7280	33,000
5	33,000	1980	5	5000	6980	28,000
6	28,000	1680	5	5000	6680	23,000
7	23,000	1380	5	5000	6380	18,000
8	18,000	1080	6	6000	7080	12,000
9	12,000	720	6	6000	6720	6,000
10	6,000	360	6	6000	6360	0
		17760	50	50000	67760	

The average yearly payment is seen to be \$6776, a little less than the guiding figure \$6793.

Many bonds pay interest twice a year, the interest rate being nominal. In such a case bonds may be called every six months instead of at periods of a year. A schedule for retirement can be calculated according to the general plan followed in the example above. But it is evidently a little early to begin to retire bonds six months after they have been sold,* and retirement is more likely to be deferred in accordance with ideas discussed in the next section.

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* It is not implied that short term bonds are not often sold. The United States government, for instance, makes many such issues, and at very low interest rates. They are taken up mostly by banks. They are, however, not callable bonds.

51. Deferred amortization of a bonded debt. The period over which bonds are subject to call may be deferred; that is, the bonds may be subject to call after the expiration of 5 years, or 10 years, or some other period. During such a period the interest on the entire issue must be paid. It is clear, however, that it is desirable to create a sinking fund during this period, in addition to paying interest, the last payment into the sinking fund being made one year before the first bonds are called for retirement. The sinking fund will itself afford an annuity extending over the period when the bonds are being called, and will lessen the amount that must be taken from earnings during the last years for the purpose of retiring the issue.

The amount to be paid into the sinking fund may be arbitrarily set, or it may be so determined that the combined interest and sinking fund payment during the first years is approximately the same as the amount spent during the later years when the bonds are being retired. The question is taken up in the exercises that follow. In Exercise 3, the amount paid into the sinking fund is arbitrarily set. In Exercise 4 the total payment over the entire period is kept essentially constant Exercise 5 is the solution for this problem in a general rather than a special case. The problem gives rise to a very instructive equation of value.

EXERCISES

1. Construct a schedule for retiring in 12 years 100 bonds each of $100 denomination, paying $2\frac{1}{2}\%$ interest, the bonds being retired at par.

2. The sum of $50,000 has been raised by selling 50 bonds each of $1000 denomination. The bonds pay 5% and are to be retired at $1030 by call over a period of 10 years. Construct a schedule.

Suggestion: If the bonds were redeemed at par, the figure to guide the yearly outlay would be

$$\$50{,}000 \frac{1}{a_{\overline{10}|}(5\%)} = \$6475.23.$$

A total of $\$30 \times 50 = \$1500$ must be paid on account of the premium, or a yearly average of $150. Thus the final guiding amount is $\$6475 + \$150 = \$6625$. Show that if the bonds are retired according to the scheme in the example in section 50 the average yearly payment is $6630.

3. The sum of \$100,000 has been raised by selling 100 bonds each of \$1000 denomination. The bonds pay 5% interest and are subject to call over a period of 10 years, the first bonds to be retired in 11 years. In addition to paying interest the first 10 years, let a sum of \$4000 be put each year into a sinking fund on which 4% interest is paid. Show that the sinking fund will provide each year

$$\$4000\,\frac{s_{\overline{10}|}(4\%)}{a_{\overline{10}|}(4\%)} = \$5920.97$$

towards retiring the bonds during the last 10 years. Construct a schedule for the retirement of the bonds, so that the yearly charge during the last years will be approximately

$$\$100{,}000\,\frac{1}{a_{\overline{10}|}(5\%)} - \$5920.97 = \$7029.49.$$

4. Bonds and retirement provision as in Exercise 3. It is again desired to use a sinking fund earning 4% for the first 10 years, the payment into the fund being determined so that the total annual payment over the entire period will be practically constant. Let R be the annual payment into the sinking fund, and show that the equation of condition is

$$5000 + R = 100{,}000\cdot\frac{1}{a_{\overline{10}|}(5\%)} - R\cdot\frac{s_{\overline{10}|}(4\%)}{a_{\overline{10}|}(4\%)},$$

so that

$$R\left(1 + \frac{s_{\overline{10}|}(4\%)}{a_{\overline{10}|}(4\%)}\right) = 7950.46,$$

and finally $R = 3205$. Thus the total annual charge over the 20-year period is approximately \$5000 + \$3205 = \$8205.

5. Let A be the amount of a bonded debt paying interest at rate i and subject to call over a period of n years, deferred m years. During the first m years, in addition to paying interest, let a sinking fund be created, into which the annual payment R is made. Suppose the sinking fund earns interest rate i', and let R be determined so that the total yearly outlay over the entire life of the bonds is maintained essentially constant. Show that the equation for R is

$$iA + R = \frac{A}{a_{\overline{n}|}(\text{rate } i)} - R\cdot\frac{s_{\overline{m}|}(\text{rate } i')}{a_{\overline{n}|}(\text{rate } i')}.$$

Thus

$$R\left(1 + \frac{s_{\overline{m}|}(\text{rate } i')}{a_{\overline{n}|}(\text{rate } i')}\right) = A\cdot\frac{1}{a_{\overline{n}|}(\text{rate } i)} - iA = \frac{A}{s_{\overline{n}|}(\text{rate } i)},$$

so that*

$$R = \frac{A\cdot a_{\overline{n}|i'}}{s_{\overline{n}|i}(a_{\overline{n}|i'} + s_{\overline{m}|i'})}.$$

The student should observe how use was made of the important relation (4). In solving a special case, as in Exercise 4, one may not detect that the relation can be employed, but when the solution for the general problem is written, its appropriateness becomes manifest.

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* In the final result the manner of indicating the interest rate in the s and a symbols has been changed, in order to condense writing.

6. A bonded debt of \$200,000 is to be retired during 10 years, deferred 15 years. The bonds pay 6% interest, interest semiannually. A sinking fund is to be accumulated over the first 15 years, the fund earning 4% nominal, converted semiannually. How much should be paid into the sinking fund every 6 months in order to keep the total semiannual payment approximately the same over the entire 25 years?

SUGGESTION: Since interest rates are nominal, conversion periods take the place of years, so that $n = 20$, $m = 30$, $i = .03$, $i' = .02$. Show that $R =$ \$2138.

7. In the situation given in Exercise 6, show that the sinking fund will provide a semiannual yield of approximately \$5304 during the 10 years when the bonds are being retired. By starting a schedule, find the number of bonds to be retired on each interest date during the first 2 years of the retirement period, if all the bonds are of \$500 denomination.

CHAPTER VII

BONDS

52. Savings bonds. In recent years, people in general have become familiar with a new type of bond, the Savings Bonds of the United States Government. For example, a bond bought for \$75 is redeemed in 10 years for \$100, or sooner for smaller amounts. The characteristic feature of such bonds is that a person's earning on his investment is paid all at one time—when he surrenders the bond. Questions connected with savings bonds are handled readily by the fundamental compound interest formula, as illustrated by Exercise 13, page 23, and Exercise 13, page 25. On account of the instructive nature of the calculation involved, we shall evaluate a savings bond at the end of each year of its life, assuming a constant effective yearly rate.

Consider a bond bought for \$75 and redeemable for \$100 in 10 years. Let i be the effective interest rate; then by (1), page 15,

$$75(1 + i)^{10} = 100,$$

so that

$$1 + i = \left(\frac{4}{3}\right)^{\frac{1}{10}}.$$

The value of the bond at the end of n years is

$$75(1 + i)^{n} = 75\left(\frac{4}{3}\right)^{\frac{n}{10}},$$

and to construct a schedule of yearly values, the quantity on the right must be evaluated for $n = 1, 2, 3 \cdots, 9$, and this is rapidly done by logarithms, noting that

$$\log 75\left(\frac{4}{3}\right)^{\frac{n}{10}} = \log 75 + n\left(\frac{1}{10}\log\frac{4}{3}\right).$$

It is seen that the logarithm of the value at the end of any year is found by adding $\frac{1}{10}\log\frac{4}{3}$ to the logarithm of the value at the end of the preceding year. The successive addition of this quantity is readily carried on by writing it on the bottom of a slip of paper and then holding it above the number to which it is to be added. In the calculation below all numbers used are shown, except 0.01 2494 on the bottom of the slip of paper, and all additions were of contiguous numbers.

$$\begin{aligned}
\log 4 &= 0.60\ 2060 \\
\log 3 &= 0.47\ 7121 \\
\hline
\log 4/3 &= 0.12\ 4939 \\
\frac{1}{10}\log 4/3 &= 0.01\ 2494 \\
\log 75 &= 1.87\ 5061 \\
\hline
\end{aligned}$$

log value 1st year = 1.88 7555, value 1st year, $ 77.19
log value 2nd year = 1.90 0049, value 2nd year, 79.44
log value 3rd year = 1.91 2543, value 3rd year, 81.76
log value 4th year = 1.92 5037, value 4th year, 84.15
log value 5th year = 1.93 7531, value 5th year, 86.60
log value 6th year = 1.95 0025, value 6th year, 89.13
log value 7th year = 1.96 2519, value 7th year, 91.73
log value 8th year = 1.97 5013, value 8th year, 94.41
log value 9th year = 1.98 7507, value 9th year, 97.16
log value 10th year = 2.00 0001, value 10th year, 100.00

It is to be noted that all logarithms were checked by the known correct value for the 10th year. Let the student verify that the interest rate is approximately 2.92%.

In reality the U. S. Savings bonds have surrender values at the end of every 6 months. The values are, however, not those that result from a constant rate (and which could be found in a way similar to the values obtained above), but are set arbitrarily, increasing slowly at first, then more rapidly, so as to encourage the holding of the bond until maturity.

53. Coupon bonds. The traditional type of bond pays the owner interest at regular periods, in most cases twice a year. Usually the bond has coupons attached, which the owner can detach on the dates shown and present to his bank, which acts as agent for him.

Bonds are classified as unregistered or registered. An unregistered bond does not show the name of the owner, and it is a very liquid form of property, that can be sold or traded readily. A registered bond on the other hand bears the owner's name, and can be transferred to another person only by proper endorsement on the bond. Some bonds are also registered as to interest; in which case they do not bear coupons, the interest being paid on the proper dates by checks sent by the issuer to the owner. Many bonds of corporations are secured by mortgages on property; bonds not so secured are customarily called *debenture* bonds. The bonds of any unit of government are secured only by its good faith and financial integrity. The sum named in the bond is called its *face value*, its *par value*, or its *denomination*. Thus a bond is described as a $100 bond, a $500 bond, or a $1000 bond. Such face value is the amount that is generally paid when the bond is finally taken up by the party that issued it. A bond may, however, have a *redemption value* greater than its face or par value. This is especially likely to be the case if the bonds are subject to redemption at call, instead of on a specified date. But even if the bond is to be paid in 10 years, 20 years, or some other definite period, such an inducement may be offered in order to facilitate the original sale.

The *bond interest rate* is merely the interest the bond pays, and a bond is described as a 5% bond, or a 6% bond, and so on. The rate always has reference to the face value, not to the redemption value, when the latter exceeds the former. Thus a $1000 5% bond pays $50 a year in the form of interest. Where interest is paid more than once a year, the rate is nominal. Thus, if a $1000 bond pays $20 semiannually, it is called a 4% bond; it would still be described as a 4% bond if it paid $10 quarterly.

54. Investment rate. Bonds can be purchased not only at the time of their issue, but at all times from investment houses. The issues of large corporations and of the government are listed in the stock exchanges, and the prices at which they are selling are given on the financial pages of newspapers. The prices at which bonds can be bought depend upon the amount of money that is seeking investment, and the general economic and financial situation. During times of unusual prosperity, when there is much money and persons are able to retain desirable investments, the prices of good bond issues will rise. On the other hand, in periods of depression some persons are forced to sell bonds in order to obtain money, many bonds will be offered for sale, and the price will fall, the fall being accentuated either by the fact that funds for investment are scarce or that potential investors are very cautious.

Various important factors that influence the price of bonds cannot be taken into account mathematically. If there is uncertainty whether the interest on a bond will be paid, and doubt as to whether it will be redeemed as stipulated, the purchase of it has a great element of risk, and it becomes a speculative and not an investment bond. Naturally the price may sink to a very low level. We do not consider such bonds, and all the evaluations that will be made are based upon the assumption that the interest will be paid regularly, and the bond redeemed as specified on its face.

The problems arising in connection with bonds are best grasped by illustrations. If at any time a person pays $100 for a 3% bond redeemable at par, it is clear that he is earning 3% on his funds. If, however, he pays less than $100 for it he is earning over 3%, while if he pays more than $100 he is earning less than 3%. The generalization is simple. Two main questions now arise.

In the first place, a person can decide what rate of interest he is willing to accept on an investment, and then determine how much he will be willing to pay for a certain bond. For instance, he may know from the general financial situation that it is not possible to invest safely at over 4%, new issues of bonds not paying more than this rate at the time, and he may feel that he should be able to put his funds where they will earn this rate. Obviously, then, he should be willing to pay more than par for a 5% bond, and less than par for a 3% bond. But how much more, or how much less? His problem is to determine how much he should pay in order that the bond he buys will be an actual 4% investment for him. We shall find the solution of this problem to be relatively simple, and it leads to the fundamental bond equation.

On the other hand a person may be offered a bond for a certain definite amount; for example, he may be offered a $500 6% bond for $515, or a $500 4½% bond for $475. He then has the problem of finding the actual interest he would be earning on his investment, if he purchased either of these bonds. This problem is a little more difficult than the former one, but it is nevertheless solvable by the fundamental bond equation, and to the derivation of this we now proceed.

55. Price to produce a given yield. We assume that the bond pays interest annually, and that it is bought on an interest date, but that the purchaser does not receive the interest then due. (Or we can say that the bond is bought just after an interest date, for instance the next day.) Let

F = face value of the bond,
C = redemption price,
r = bond rate of interest,
n = number of years until redemption,
i = investment rate,
A = purchase price.

The first five quantities are assumed to be given, and the problem is to find A.

It is merely necessary to note what the bond actually does for the investor. In the first place it will bring him the sum C in n years. At the present time the value of this sum is

$$Cv^n, \qquad v = \frac{1}{1+i},$$

for he is investing his funds to produce the rate i.

In the second place he receives each year interest to the amount rF, beginning in one year and continuing for n years. These returns constitute an annuity whose present value is

$$rF \cdot a_{\overline{n}|} \quad \text{(rate } i\text{)}.$$

Thus the present value of the total return from the bond is

$$Cv^n + rF \cdot a_{\overline{n}|} \quad \text{(rate } i\text{)}.$$

This is then the amount the purchaser should pay. Therefore

$$(1) \qquad \boldsymbol{A = Cv^n + rF \cdot a_{\overline{n}|}} \quad \text{(rate } \boldsymbol{i}\text{)}.$$

This is the fundamental bond equation.

The difference

$$P = A - C$$

is called the premium paid for the bond, and it is an important quantity since it represents the difference between what the purchaser spends for the bond when he buys it and what he receives when the bond matures and he receives the redemption value. When P so defined is a negative number, the bond is said to be purchased at a discount, the amount of the discount being the premium with the negative sign dropped. Thus in practice the word *premium* is restricted to those cases where the bond is bought at more than its redemption value, while the word *discount* is used in the contrary case. But when algebraic formulas are being developed one always adheres to the general definition $P = A - C$ and uses the customary terminology in accordance with the sign of the result.

Example 1. How much should be paid for a \$500 5% bond, interest payable annually, redeemable at par in 12 years, in order that it yield 6%?

Here $F = C = 500$, $r = .05$, $rC = 25$, $i = .06$, $n = 12$.

Hence

$$\begin{aligned} A &= \$500v^{12} + \$25a_{\overline{12}|}(6\%) \\ &= \$500(.496\,969) + \$25(8.38384) \\ &= \$248.484 + \$209.596 = \$458.08. \end{aligned}$$

In this case there is a *discount* of \$41.92.

The majority of bonds pay interest twice a year. The bond rate r is then nominal as noted above; for example, a \$100 4% bond with interest payable semiannually pays \$2 interest twice a year. The investment rate is then also usually nominal with conversion twice a year. The fundamental formula (1) can still be used, for the essential elements are that r be the bond rate per period, i the investment rate per period, and n the number of periods.

Example 2. A \$100 4% bond, paying interest semiannually, redeemable in 15 years at \$105, is offered for sale. How much should be paid for it in order to yield the purchaser 3% nominal, converted semiannually?

Equivalent to $F = \$100$, $C = \$105$, $r = .02$, $i = .015$, $n = 30$.*

Hence

$$\begin{aligned} A &= \$105v^{30} + \$2a_{\overline{30}|}(1\tfrac{1}{2}\%) \\ &= \$105(.63976) + \$2(24.016) \\ &= \$67.17 + \$48.03 = \$115.20 \end{aligned}$$

A *premium* of \$10.20 would therefore be paid.

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* That is, the value is precisely that of a bond with $F = \$100$. $C = \$105$, paying an annual interest of 2%, redeemable in 30 years, and bought to yield 1½% interest annually.

It is of course a simple matter to write a formula that provides for interest payable more than once a year, and the formula will be given, but it is to be emphasized that in a fundamental way it gives nothing new.

Let

p = number of times a year that interest is paid,
j = nominal investment rate,

other letters having the meaning they had in (1). Then

$$A = Cv^{pn} + \frac{r}{p} F \cdot a_{\overline{pn}|} \quad \left(\text{rate } \frac{j}{p}\right). \tag{2}$$

For the usual case $p = 2$,

$$A = Cv^{2n} + \frac{r}{2} F \cdot a_{\overline{2n}|} \quad \left(\text{rate } \frac{j}{2}\right). \tag{3}$$

Example 3. The bond in Example 2. Here $r = .04$, $j = .03$, $n = 15$. The same equation is obtained as before.

If the bond pays interest p times a year, and the effective rate is i, we have

$$A = Cv^n + rF \cdot a_{\overline{n}|} \cdot \frac{i}{j_{(p)}} \quad (\text{rate } i), \tag{4}$$

as is seen by referring to (13), page 69.

Example 4. A 6% bond with face value \$1000, paying interest semiannually, redeemable at par in 8 years, is offered for sale. How much is it worth on a 5½% effective basis?

$$A = \$1000v^8 + \$60a_{\overline{8}|}\frac{i}{j_{(2)}} \; (5\tfrac{1}{2}\%)$$
$$= \$651.599 + \$385.230 = \$1036.83.$$

$$\begin{aligned} \log 60 &= 1.77\,8151 \\ \log a_{\overline{8}|}(5\tfrac{1}{2}\%) &= 0.80\,1717 \\ \log \frac{i}{j_{(2)}}\,(5\tfrac{1}{2}\%) &= 0.00\,5852 \\ \hline &\ 2.58\,5720 \end{aligned}$$

A premium of \$36.83 would therefore be paid.

56. Calculation of the premium. A simple formula for the direct calculation of the premium exists in the case $C = F$, and can be used to evaluate a bond. Let the premium be P, then

$$P = A - F.$$

By (2),

$$P = Fv^{pn} - F + \frac{r}{p} F \cdot a_{\overline{pn}|} \quad \left(\text{rate } \frac{j}{p}\right)$$

$$= F(v^{pn} - 1) + \frac{r}{p} F \cdot a_{\overline{pn}|}.$$

But by (8), page 50,

$$a_{\overline{pn}|} = \frac{1 - v^{pn}}{\dfrac{j}{p}},$$

so that

$$v^{pn} - 1 = -\frac{j}{p} a_{\overline{pn}|}.$$

Substituting in the equation above, we find

$$P = F\left(\frac{r}{p} - \frac{j}{p}\right) \cdot a_{\overline{pn}|} \quad \left(\text{rate } \frac{j}{p}\right). \tag{5}$$

Thus the premium is merely the present value of the interest payments on a bond whose interest rate is the difference between the actual bond rate and the investment rate. This result could have been anticipated.

EXERCISES

1. Let the bond rate be r, interest payable p times a year, and let the investment rate be $j_{(p)}$. Show that

$$P = (C - F)(v^{np} - 1) + F\left(\frac{r}{p} - \frac{j}{p}\right) \cdot a_{\overline{np}|} \left(\text{rate } \frac{j}{p}\right).$$

2. Find the purchase prices of the following bonds, and also the premium or discount at which they would be bought.*

	Face Value	Redemption Value	Years to Maturity	Bond Rate	Interest Payable	Investment Rate
A (1)	\$1000	\$1000	9	6%	Annually	$i = 5\frac{1}{2}\%$
(2)	500	510	6½	4%	Semiannually	$j_{(2)} = 5\%$
(3)	100	105	12	4½%	Semiannually	$j_{(2)} = 3\%$
(4)	1000	1000	10	5½%	Semiannually	$i = 5\%$
(5)	500	500	8	5%	Semiannually	$j_{(2)} = 4\%$
(6)	100	110	15	5%	Quarterly	$j_{(4)} = 4\%$
B (1)	500	500	12	5%	Annually	$i = 6\%$
(2)	1000	1050	9½	5%	Semiannually	$j_{(2)} = 4\%$
(3)	500	510	10	5¼%	Semiannually	$j_{(2)} = 6\%$
(4)	100	100	6	5½%	Semiannually	$i = 5\%$
(5)	1000	1000	15	4¾%	Semiannually	$j_{(2)} = 2\frac{1}{2}\%$
(6)	500	500	20	3%	Quarterly	$j_{(4)} = 2\%$

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* The A and the B group of exercises are combined in one table.

57. Bonds bought between interest dates. It is clear that in actual practice bonds will not generally be purchased just after an interest date, as we have assumed. When a bond is bought between interest dates, part of the interest earned since the last interest date belongs to the seller, and part to the buyer. There are different ways in which one can determine the relative amounts, using a sound basis for the procedure. We shall consider only the simplest methods.

Let A_0 be the value of the bond just after the last interest payment, as found from (1), (2), or (4). Let t be the fraction of the year * that has elapsed. If the purchaser had bought the bond on the last interest date he would consider its value on the new day as

$$A = A_0(1+i)^t \tag{6}$$

or

$$A = A_0(1+ti), \tag{7}$$

according as he used compound or simple interest. We can then, with much reason, regard either of these as the amount he would be willing to offer for the bond.

Example. A \$1000 $5\frac{1}{2}\%$ bond paying interest annually on January 1, and redeemable at par on January 1, 1940, was bought May 1, 1928, at a price to yield the investor 6%. How much should the investor have offered?

Since on January 1, 1928, 12 years remained until maturity, the price then would have been

$$\begin{aligned} A_0 &= \$1000v^{12} + \$55a_{\overline{12|}}(6\%) \\ &= \$496.969 + \$461.111 = \$958.08. \end{aligned}$$

On May 1, 1928, the price would be

$$\$958.08(1.06)^{1/3} = \$958.08(1.019\ 613) = \$976.87$$

if compound interest is used, and

$$\$958.08(1.02) = \$977.24$$

if simple interest is used.

There is another method of reasoning that will lead to (7), although at first the identity of the two methods might not be suspected.

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* In the following formulas, if the investment rate is the nominal rate $j_{(p)}$, the interest i should be replaced by j/p and t should be the fraction of the corresponding period.

Let A_0 be the value just after the last interest payment, which is assumed to be n years before the redemption of the bond, and A_1 the value just after the next payment. The value just before the next payment will be A_1 increased by the interest then due, and will therefore be

$$A_1 + rF.$$

Thus the value of the bond increases by the amount

$$A_1 + rF - A_0.$$

If we assume that the increase is uniform, we shall have as the desired value

$$A = A_0 + t(A_1 + rF - A_0). \tag{8}$$

Now (8) will reduce to (7) and the demonstration is an instructive exercise in the use of the various investment symbols.

We have

$$\begin{aligned} A_0 &= Cv^n + rF \cdot a_{\overline{n}|}, \\ A_1 &= Cv^{n-1} + rF \cdot a_{\overline{n-1}|}. \end{aligned}$$

Hence

$$\begin{aligned} A_1 + rF - A_0 &= Cv^{n-1} + rF \cdot a_{\overline{n-1}|} + rF - Cv^n - rF \cdot a_{\overline{n}|} \\ &= C(v^{n-1} - v^n) + rF(1 + a_{\overline{n-1}|} - a_{\overline{n}|}). \end{aligned}$$

But

$$v^{n-1} - v^n = v^n\left(\frac{1}{v} - 1\right) = v^n(1 + i - 1) = iv^n,$$

and

$$\begin{aligned} 1 + a_{\overline{n-1}|} - a_{\overline{n}|} &= 1 + \frac{1 - v^{n-1}}{i} - \frac{1 - v^n}{i} \\ &= 1 + \frac{v^n - v^{n-1}}{i} = 1 + \frac{v^n}{i}\left(1 - \frac{1}{v}\right) \\ &= 1 - v^n = i \cdot a_{\overline{n}|}. \end{aligned}$$

Hence

$$A_1 + rF - A_0 = Civ^n + irF \cdot a_{\overline{n}|} = iA_0,$$

so that by (8)

$$A = A_0 + tiA_0 = A_0(1 + ti),$$

which is the value given in (7).

EXERCISES

Find the purchase prices of the following bonds, using equation (7), all the bonds being redeemable at par.

	Face Value	Bond Rate	Interest Paid	Redemption Date	Date Bought	Investment Rate
1	\$ 500	5%	Jan. 1	Jan. 1, 1935	May 1, 1927	$i = 4\%$
2	1000	4½%	March 1 and Sept. 1	Sept. 1, 1950	July 10, 1942	$j_{(2)} = 3\frac{1}{2}\%$
3	100	4½%	April 15 and Oct. 15	April 15, 1942	Jan. 1, 1933	$j_{(2)} = 6\%$
4	500	5%	Jan. 1 and July 1	July 1, 1955	April 10, 1938	$j_{(2)} = 4\%$

58. Bond tables. By means of the fundamental bond formula it is possible to construct a table that will give the purchase price of bonds for different periods to maturity, different bond rates, and bought to give different investment rates. Such tables exist.* In order to provide for the necessary range in data the tables are quite extensive, and are not included in such a book as the present one. Bond tables assume that the bond is redeemable at its face value. If this is not the case it is necessary to add to the value given in the table the present value of the excess of the redemption price over the face value, this present value being

$$(C - F)v^{2n}, \quad \left(\text{rate } \frac{j}{2}\right).$$

In case a table of investment functions is not available, it is possible to determine the quantity v^{2n} very easily from a bond table, as shown in Exercise 2, page 109.

A small extract from a bond table follows.†

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* An extensive table is that of C. E. Sprague, *Extended Bond Tables*, 4th ed., The Ronald Press Co., N. Y., 1927. It has for its base a bond of \$1,000,000, and so is applicable to large amounts.

† Adapted with permission from the tables of Rollins, The Financial Publishing Co., Boston.

VALUE OF A $100 BOND DUE IN 10 YEARS, INTEREST PAYABLE SEMIANNUALLY

Bond Rate / Investment Rate	3%	3½%	4%	4½%	5%	6%	7%
4.00	91.82	95.91	100.00	104.09	108.18	116.35	124.53
4.10	91.05	95.12	99.19	103.25	107.32	115.46	123.60
4.20	90.28	94.33	98.38	102.43	106.48	115.58	122.67
4.30	89.52	93.55	97.58	101.61	105.64	113.70	121.76
4.40	88.77	92.78	96.79	100.80	104.81	112.83	120.85
4.50	88.03	92.02	96.01	100.00	103.99	111.97	119.95
4.60	87.29	91.26	95.23	99.21	103.18	111.12	119.07
4.70	86.56	90.51	94.47	98.42	102.37	110.28	118.18
4.80	85.84	89.77	93.71	97.64	101.57	109.44	117.31
4.90	85.12	89.04	92.95	96.87	100.78	108.61	116.45
5.00	84.41	88.31	92.21	96.10	100.00	107.79	115.59

It is seen that the differences in the columns in the table are essentially constant so that simple interpolation will give accurate results.

EXERCISES

From the preceding bond table find the value of the following bonds, all paying interest semiannually.

	Face Value	Bond Rate	Due	Redemption Value	To Yield
1	$ 500	6%	10 years	par	4¾%
2	1000	4¾%	10 years	par	4%
3	1000	4¾%	10 years	par	4¼%
4	100	3½%	10 years	110	4½%
5	500	5%	$9^{10}/_{12}$ years	par	4%

59. Amortization of premium. When a bond is bought at a premium, the buyer will at the time of maturity receive less than he paid for the bond. As a result his capital will be lessened, unless he takes measures to prevent it. Especially in the case of bonds bought by trustees of estates, or trustees of an institution, such as a college, the original investment should be re-established. The premium owes its origin to the fact that the bond rate is greater than the investment rate. It is clear then that a part of the interest received should be regularly saved, the part being the excess of the interest actually received over what the interest would be at the investment rate. In this way some of the premium is restored at each interest date, and as a result the amount of money that is represented by the bond at any time steadily decreases. That is, the *book value* of the bond diminishes, the book value at any time plus what has been saved out of the interest to replace the premium being always equal to the original purchase price of the bond.

As an example, suppose that on January 1, 1923, a \$1000 bond that pays 6%, interest semiannually, and matures in 4 years is bought to yield 5%. The purchase price is

$$A = \$1000\,v^8 + \$30 \cdot a_{\overline{8|}}(2\tfrac{1}{2}\%) = \$1035.85.$$

At the end of 6 months, interest to the amount of \$30 is received, but as the bond was bought to yield 5% nominal, the interest at that rate on the invested capital is

$$\$1035.85(.025) = \$25.90.$$

The difference \$30 − \$25.90 = \$4.10 must be regarded as a portion of the premium of \$35.85, and should be saved. The capital invested in the bond for the next 6 months is then \$1035.85 − \$4.10 = \$1031.75. One continues as shown in the table. The interest on the book value for one date is computed from the book value for the preceding date. The amount available for amortization is then subtracted from the book value for the preceding date in order to obtain the book value for the new date.

Date	Book Value	Interest on Book Value at 2½%	Income from Bond	For Amortization
Jan. 1, 1923	\$1035.85			
July 1	1031.75	\$25.90	\$30	\$4.10
Jan. 1, 1924	1027.54	25.79	30	4.21
July 1	1023.23	25.69	30	4.31
Jan. 1, 1925	1018.81	25.58	30	4.42
July 1	1014.28	25.47	30	4.53
Jan. 1, 1926	1009.62	25.36	30	4.64
July 1	1004.86	25.24	30	4.76
Jan. 1, 1927	1000.00	25.12	30	4.88
		\$204.15	240	35.85

It is seen that the total amount for amortization, namely $35.85, is exactly equal to the premium paid on the bond. It is also to be observed that the sums taken out for amortization are not put at interest, but are merely saved. They are of course available for new investment.

60. Accumulation of the discount. If a bond is bought at a discount, the interest received from the bond is not as great as the interest on the purchase price at the investment rate. The difference must be regarded as going into an increased value of the bond, which steadily advances. In such a case the purchaser of the bond is not only living within earned income when he spends all he receives in interest, but he is also automatically saving some of his earnings and increasing his capital.

A schedule to show how the value of the bond increases from one interest date to another is easily constructed in a manner similar to that used in the last paragraph.

Example. A $500 5% bond, interest April 15 and October 15, maturing October 15, 1930, was bought April 15, 1927, at a price to yield 6% nominal converted semiannually. Construct a schedule showing the increase in the book value of the bond.

The purchase price was

$$A = \$500v^7 + \$12.50 \cdot a_{\overline{7}|}(3\%) = \$484.42.$$

The schedule follows:

DATE	BOOK VALUE	INTEREST ON BOOK VALUE	INCOME FROM BOND	INCREASE OF BOOK VALUE
Apr. 15, 1927	$484.42			
Oct. 15	486.45	$14.53	$12.50	$2.03
Apr. 15, 1928	488.54	14.59	12.50	2.09
Oct. 15	490.70	14.66	12.50	2.16
Apr. 15, 1929	492.92	14.72	12.50	2.22
Oct. 15	495.21	14.79	12.50	2.29
Apr. 15, 1930	497.57	14.86	12.50	2.36
Oct. 15	500.00	14.93	12.50	2.43
		103.08	87.50	15.58

If on any interest date the owner sold the bond for less than the book value the bond would have been a less profitable investment than he had intended; if he sold it for more, he would have received a higher rate of interest than the contemplated 6%.

61. Remarks. The last two sections show that great care is necessary in keeping an accurate account of funds that are invested in bonds. The small investor of personal savings may not use the accurate procedure that has been explained, but some simpler "straight line method."

When the straight line method is used and a bond is bought at a premium, the book value is decreased by the same amount at each interest date. Thus if a premium of \$1200 has been paid, and there are 20 periods, the book value will be decreased \$60 each interest period. The objection to the method is clear. Each period the same amount of income must be set aside for amortization of the premium. The remaining expendable income will be constant, and as the book value diminishes, the percentage net yield of the bond increases; that is, it does not remain constant.

If a bond is bought at a discount and the straight line method is used, the book value is increased a constant amount each period. In this case, if the bond is sold before maturity at the book value that has been set down, a greater amount will be received than would be received if the method of the last section had been followed for obtaining the book value.

EXERCISES*

A

1. A \$10,000 4½% bond, paying interest March 15 and September 15, redeemable September 15, 1930, was bought March 15, 1922, at a price to yield 3½%. Make a schedule showing the book value of the bond up to September 15, 1925. In an audit at that time what liability should be shown toward the amortization of the premium?

2. Construct a schedule showing the book value of a \$1000 bond, paying 5%, interest January 1, and July 1, redeemable at \$1050 on January 1, 1935, bought July 1, 1931, at a price to yield 4%.

3. Bonds with face value \$50,000, paying 6%, interest April 1 and October 1, redeemable October 1, 1932, were bought April 1, 1926, at a price to yield 7%. What was the book value on October 1, 1928?

4. A \$1000 bond, paying 3½%, interest quarterly, was bought at a price to yield 4%, 4 years before maturity. For how much should it have been sold 15 months after purchase to realize the investment rate?

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* In all exercises it is assumed that the conversion period for the investment rate is the same as the period in which the bond pays interest.

5. Find the book value of the bond in Exercise 1 by the straight line method.

6. Find the book value of the bonds in Exercise 3 by the straight line method.

B

1. Bonds with face value of \$20,000, paying 6%, interest January 1 and July 1, redeemable July 1, 1929, were bought January 1, 1921, at a price to yield 5%. Make a schedule showing the book value up to July 1, 1924. In an audit at that time what liability should be shown toward the amortization of the premium?

2. Construct a schedule showing the book value of a \$10,000 bond, paying 5%, interest April 15 and October 15, redeemable at \$10,400 April 15, 1935, bought October 15, 1931, at a price to yield 7%.

3. Bonds with face value \$40,000, paying 5%, interest March 1 and September 1, payable September 1, 1934, were bought March 1, 1928, at a price to yield the investor 6%. What was the book value September 1, 1930?

4. A \$500 bond, paying 4%, interest quarterly, was bought at a price to yield 5%, 5¾ years before maturity. For how much should it have been sold 18 months after its purchase to realize the investment rate?

5. Find the book value of the bond in Exercise 1 by the straight line method.

6. Find the book value in Exercise 3 by the straight line method.

62. Yield of a bond bought at a given price. In the majority of cases an investor cannot buy a bond at a price to yield an investment rate set by himself. Usually he must make his purchases in the open market, and must pay the prices at which bonds are offered. The problem then arises of determining the actual yield. This is obviously a thing the investor should know, and without it an accurate schedule showing the book value cannot be computed.

One should first inquire as to whether the problem is readily solvable algebraically by using the fundamental bond equation. If we take the customary case $p = 2$, assume that the bond is redeemable at par, and use equation (6), we find upon substituting the value for $a_{\overline{2n}|}$ that we have an equation of degree $2\ n$ in j, which is the unknown quantity. (If the purchase price of the bond is given, the premium P is easily deduced.) Algebraically then there is difficulty, but there is a fact that simplifies the problem somewhat. The unknown j is a relatively small number; its powers diminish rapidly, and terms of higher degree can be neglected. But uncertainty is introduced if n is large, for powers of n enter the coefficients of the powers of j. It is possible to obtain an approximate value of j as the solution of an equation of the first degree. But it is not possible to put much reliance upon it, and we shall consider only tabular methods of finding j.

(1) Use of bond table. If a bond table is available, the yield is readily found by interpolation.

Example 1. What is the yield of a \$100 6% bond, payable in 10 years, bought for \$110?

From the table on page 102 we find:

Yield	Price
4.70%	\$110.28
4.80	\$109.44.

Letting j be the required yield, we have

$$\frac{j - 4.70}{4.80 - 4.70} = \frac{110.28 - 110}{110.28 - 109.44},$$

or

$$\frac{j - 4.70}{.10} = \frac{.28}{.84} = .333.$$

Hence

$$j = 4.7333\%.$$

If the bond is redeemable above par, the present value of the excess $C - F$ should be subtracted from the price of the bond before using the table. The present value should be calculated by using the investment rate, but this is the unknown. One can estimate the investment rate, or use the bond rate. After the bond rate has been found from the table, the calculation can be repeated if greater accuracy is required.

(2) Use of investment tables. Using two convenient trial values for the investment rate, the corresponding purchase prices can be found, the rates being so selected that the two purchase prices obtained "bracket" the given purchase price.

By using proportional parts the required yield rate can then be found.

Example 2. A \$500 bond paying 5%, interest semiannually, redeemable at \$520 in 10 years, has been bought for \$485. Find the investment rate.

We have

$$
\begin{aligned}
j &= 6\%, & A &= \$520v^{20} + \$12.50\cdot a_{\overline{20|}}(3\%) &&= \$473.88,\\
j &= 5\%, & A &= \$520v^{20} + \$12.50\cdot a_{\overline{20|}}(2\tfrac{1}{2}\%) &&= \$512.20.
\end{aligned}
$$

Therefore

Yield	Price
$j = 6\%$	\$473.88
$j = ?$	485.00
$j = 5$	512.02.

Hence

$$
\begin{aligned}
\frac{j-5}{6-5} &= \frac{512.02 - 485.00}{512.02 - 473.88},\\
j - 5 &= \frac{27.02}{38.14} = .708,\\
j &= 5.71\%.
\end{aligned}
$$

The accuracy of the result depends upon the width of the "bracket." If the investment table has an extensive list of interest rates, the bracket can be made small and the resulting accuracy increased.

A third way to find the yield rate of a bond is given in Exercises 3 and 6, pages 109 and 110.

EXERCISES

A

1. Find the nominal investment rates of the following bonds all paying interest semiannually.

	Face Value	Redemption Value	Years to Maturity	Bond Rate	Purchase Price
(1)	\$1000	\$1000	9½	5½%	\$952.00
(2)	500	520	7	4%	530.00
(3)	100	100	15½	4½%	84.50
(4)	500	500	$6^{10}/_{12}$	5%	525.00
(5)	10,000	10,000	22	6%	12,490.00

2. Use the bond table on page 102 to find the investment rate of a \$1000 4% bond bought for \$963.40 and redeemable at par in 10 years.

3. Using the bond table, find the yield of a \$500 5% bond bought for \$550 and redeemable in 10 years for \$540.

SUGGESTION: The present value of the \$40 excess of C over F is \$24.41 on a 5% nominal basis and is \$26.91 on a 4% nominal basis. Take \$25.50 as an approximation at the investment rate, thus giving \$524.50 as the purchase price of the bond on the basis that it is redeemable at par.

B

1. Find the nominal investment rates of the following bonds, all paying interest semiannually.

	Face Value	Redemption Value	Years to Maturity	Bond Rate	Bought for
(1)	\$100	\$110	12	3¼%	\$103.50
(2)	500	500	5½	2%	485.00
(3)	1000	1000	7⁵/₁₂	5%	1025.00
(4)	500	500	20	4%	350.00
(5)	10,000	10,000	25	6%	11,250.00

2. Use the bond table on page 102 to find the investment rate of a \$500 5% bond bought for \$525 and redeemable at par in 10 years.

ALGEBRAIC EXERCISES

1. Let $C = F$; that is, suppose the bond is redeemable at par. Let

$$K = Fv^n,$$

so that K represents the present value of the final redemption value of the bond. Using the value for $a_{\overline{n}|}$ in (1) prove that

$$P = (F - K)\left(\frac{r}{i} - 1\right).$$

2. Let A' be the value of the bond as given by (3) with r replaced by r'. Eliminate $a_{\overline{2n}|}$ between the two equations and show that if $r' = r + .01$, and $F = 100$, then

$$v^{2n} = \frac{A}{100} - r(A' - A), \text{ rate } j/2.$$

This relation gives a method to obtain v^{2n} from a bond table, and therefore a way to find the present value of $C - F$ when the bond is not redeemable at face value. Use the bond table on page 102 to find v^{20} for 2% and 2½%.

3. Assume $C = F$ and in (3) put $A = F + P$, where P is the premium. Also replace v^{2n} by its value in terms of $a_{\overline{2n}|}$ and $j/2$, and show that

$$j = r - \frac{2P}{Fa_{\overline{2n}|}}, \quad a_{\overline{2n}|} \text{ at rate } j/2.$$

This formula gives a method to calculate the investment rate of a bond by "iteration." Take $1/a_{\overline{2n}|}$ at an assumed investment rate, and calculate j. Repeat, evaluating $1/a_{\overline{2n}|}$ for tabular value nearest value just found for $j/2$, getting a new value for j, and so on. The second or third value should suffice. The method will be illustrated in some extreme cases.

4. A $100 7% bond, interest semiannually, redeemable in 20 years, was bought for $160. Find the investment rate.

SOLUTION: The investment rate was low. Use $1/a_{\overline{40}|}$ for 2%, and obtain $j = .07 - 1.2(.0365) = .026$. Now use $1/a_{\overline{40}|}$ 1½%, giving $j = .07 - 1.2(.0334) = .02992$, so that the investment rate was 2.992%.

5. A $100 6% bond, interest semiannually, redeemable in 20 years, was bought for $66. What was the investment rate?

6. In case $C \neq F$, show that

$$j = r\frac{F}{C} - \frac{2P}{C}\frac{1}{a_{\overline{2n}|}},\quad a_{\overline{2n}|} \text{ at rate } j/2,$$

and explain how this formula can be used to find the yield rate on a bond with bond rate r, paying interest semiannually, redeemable above par, and bought either at a premium or a discount.

7. A $100 5% bond paying interest semiannually, redeemable in 12 years at $110, was bought for $140. Find the investment rate.

8. A $1000 4% bond paying interest semiannually, redeemable in 8 years at $1050 was bought for $850. Find the investment rate.

CHAPTER VIII

DEPRECIATION AND REPLACEMENT

63. Introduction. One of the most important considerations in the successful and economical conduct of a business is that of sound planning for repairs and replacements. Buildings and other properties deteriorate and must be replaced. It is imperative that costs for keeping properties of all kinds in a satisfactory and productive condition be spread as uniformly as possible over different operating years, so that possible embarrassing situations may be avoided. The problem is a difficult one involving the estimation of future costs, which are predictable only with considerable uncertainty. No plan may work out perfectly, but nevertheless there are sound methods that can be followed to obtain as good a solution as the uncertainties in the problem will allow.

In the case of a large concern that has careful audits made and statements published for the benefit of stockholders, there arise perplexing questions as to accounting. Thus in the list of assets of a company the statement that \$300,000 is held for depreciation may give barely a hint of what is involved. A multitude of considerations may have entered, requiring extensive calculations as well as very careful bookkeeping.

64. Straight line method of depreciation. This, the simplest way of handling depreciation, is similar to the straight line method for the amortization of the premium of a bond, which was described in the last chapter.

Let

C = cost of the article considered,
n = probable life in years,
S = scrap value,
D = annual depreciation charge.

It is customary to speak of the *wearing value* W, where by definition

$$W = C - S.$$

In the straight line method we put

$$D = \frac{C - S}{n}. \tag{1}$$

Then each year the book value is decreased by the constant amount D. Consequently the amount D should be saved each year out of earnings and put in a replacement account; that is, either in a bank or in liquid and conservative securities.

A table showing the book value and the amount that has been set aside as depreciation charges is very easily made.

65. Sinking fund method. The amounts set aside for depreciation may earn a moderate rate of interest, as when they are deposited in a savings account in a bank.*

Let i be the rate of interest earned by the sinking fund. It is clear that we have for the annual charge.

$$D = \frac{W}{s_{\overline{n}|}}. \tag{2}$$

In order to have the book value of the article at the end of any year, it is necessary to know how much is in the sinking fund; the latter in this case will not be a simple multiple of D as in the straight line method. If we denote the amount in the fund at the end of r years by R_r, we have

$$R_r = D \cdot s_{\overline{r}|} = \frac{W}{s_{\overline{n}|}} \cdot s_{\overline{r}|}. \tag{3}$$

Example. Construct a table showing the book value and the replacement account for a truck costing $1200, having a probable life of 5 years and a scrap value of $100, if a sinking fund earning 3% is used.

We have

$$W = \$1200 - \$100 = \$1100, \qquad D = \$1100 \frac{1}{s_{\overline{5}|}(3\%)} = \$207.19.$$

To find the amounts in the sinking fund at the end of the 2nd, 3rd, and 4th years, we write log $s_{\overline{2}|}$, log $s_{\overline{3}|}$, and log $s_{\overline{4}|}$ in a line, write log D on the bottom of a piece of paper, and holding it over the preceding numbers one after the other, add it to each of them, obtaining log R_2, log R_3, and log R_4. The quantities R_2, R_3, R_4 are written in their proper places in the table.

---

* Instead of being invested outside the business itself, sinking funds may be used to expand the business. In such a case it is usual to think of the funds as earning a higher rate of interest. Evidently many new questions are raised. Paton and Stevenson, *Principles of Accounting* (The Macmillan Co., N. Y., 1919), pp. 503–505.

$$\log D = 2.31\ 6369$$

r	2	3	4
$\log s_{\overline{r}\vert}(3\%)$	0.30 7496	0.49 0085	0.62 1553
$\log R_r$	2.62 3865	2.80 6454	2.93 7922

Age in Years	Replacement Charge	Total in Replacement Fund	Book Value
0			$1200.00
1	$207.19	$207.19	992.81
2	207.19	420.60	779.40
3	207.19	640.40	559.60
4	207.19	866.81	333.19
5	207.19	1100.00	100.00

66. Remarks. When a sinking fund is used, interest is earned on that part of the original investment that has been placed in the fund. This raises the question as to whether interest should also be charged on the entire investment. In many cases it should certainly be taken into account in making an analysis of the profits that are left after the sinking fund and other charges have been met.

Example. The owner of the truck considered in the preceding example is assumed to have started with a capital of $1200. At 6% he should receive $72 a year from his capital alone. Thus after putting $207.19 into the sinking fund at the end of the first year, he should consider $72 of the remaining profit as income from his investment, and the balance as earned by his work at operating the truck. For the next year the interest that he might put aside would be 6% on the new book value and 3% on what was in the sinking fund, since an earning of 3% on the sinking fund is being automatically provided for.

The next table shows interest on the basis described, book values and sinking fund amounts being from the preceding table. (The new table would be added as new columns to the former table, if the new interest figures were desired.) The replacement fund and book values for one year are the principals used in calculating interest for the next year.

Age in Years	2% on Replacement Fund	6% on Book Value	Total Interest
1		$72.00	$72.00
2	$6.22	59.57	65.79
3	12.62	46.76	59.38
4	19.21	33.58	52.79
5	26.00	19.99	45.99

There are cases where no attempt can well be made to fix beforehand the profit to be earned on investment. Thus suppose that a factory is built and equipped at an outlay of \$1,000,000, raised by selling common stock. From the income all salaries and wages are paid, current operating expenses are met, and also sinking fund charges to provide for repairs and replacements. What is left is the net profit on the investment, which varies greatly from year to year, causing the fluctuation in the value of the stock. Careful management will during good years not divide all the profit, but will put some of it into a surplus fund, which can be drawn upon in years when earnings are low.

The question of when interest should be computed on remaining book values, the rate to be used, the accounting procedure to be employed, and so forth, is one on which there can be much debate. It is described in works dealing with advanced accounting.

67. Constant percentage method. Instead of reducing the book value by the same amount each year or using a sinking fund, the book value is often reduced each year by a certain per cent. Let x be the rate used in reducing the book value. Then the original book value C is reduced the first year by the amount $x \cdot C$, so that the book value at the end of the first year is $(1 - x)C$. At the end of the second year it is $(1 - x)^2C$, and so on. The book value must ultimately reduce to the scrap value. Thus x must be determined by the equation

$$(1 - x)^n C = S. \tag{4}$$

Hence

$$1 - x = \left(\frac{S}{C}\right)^{1/n}, \tag{5}$$

and

$$x = 1 - \left(\frac{S}{C}\right)^{1/n}. \tag{6}$$

The book value at the end of r years is

$$R_r = C(1 - x)^r. \tag{7}$$

In this case the book value is first computed each year; from comparison with the preceding year the amount to be set aside for depreciation is easily obtained. The sinking fund may, of course, be productively employed, but any earning from it is not shown in the account for the article.

Example. Using the constant percentage method, make a table showing book value and depreciation for the truck in the example on page 112.

We have

$$1 - x = \left(\frac{100}{1200}\right)^{1/5} = \frac{1}{(12)^{1/5}}.$$

$$\begin{aligned} \text{colog } 12 &= 8.92\ 0819 \\ \log(1-x) = \tfrac{1}{5} \text{ colog } 12 &= 9.78\ 4164 \\ \log C &= 3.07\ 9181 \end{aligned}$$

r	1	2	3	4
$\log(1-x)^r$	9.78 4164	9.56 8328	9.35 2492	9.13 6656
$\log R_r$	2.86 3345	2.64 7509	2.43 1673	2.21 5837

Age in Years	Book Value	Depreciation for Year	Total Depreciation
0	$1200.00		
1	730.04	$469.96	$ 469.96
2	444.13	285.91	755.87
3	270.19	173.94	929.81
4	164.38	105.81	1035.62
5	100.00	64.38	1100.00

We have $1 - x = .6084$, $x = .3916$. That is, 39.16% of the book value is written off each year. In the logarithmic computation of the table it is not necessary to have this figure.

EXERCISES

A

1. A machine costing $8000 has a probable life of 15 years and an estimated scrap value of $300. Using the straight line method, find the book value at the end of 5 years, and the amount that should be carried in the depreciation account at the end of 7 years.

2. A taxicab company purchased 20 cabs costing $1200 each. The estimated life of a cab was 4 years, and the scrap value was $200. Construct a schedule showing book values and depreciation account if a sinking fund earning 2½% was used.

3. A machine costing $12,000 must be replaced in 10 years, when its scrap value will be $250. Using the constant percentage method, find the book values at the end of each of the first 4 years and also the amounts in the depreciation account. What percentage of the book value is written off each year?

4. A plant consists of 3 parts, with costs, scrap values, and probable lives as follows:

Part	Cost	Scrap Value	Life
A	$75,000	$15,000	30 years
B	30,000	4,000	18 years
C	10,000	500	10 years

Replacement is to be provided for by a sinking fund earning 3% annually. How much will be in the fund at the end of 8 years?

5. A building that cost $40,000 has a probable life of 25 years and a scrap value of $5000. A sinking fund into which semiannual payments are made has been created, the fund earning interest at the rate $j_{(2)} = 3\%$. What will be the book value of the building at the end of 15 years?

6. If the constant percentage method were used, what would be the value of the building in Exercise 5? Show that the result is independent of the frequency with which the book value is determined each year.

7. In the constant percentage method, how can complete deterioration be provided for?

B

1. A machine costing $12,000 has a probable life of 10 years and an estimated scrap value of $300. Using the straight line method, find the book value at the end of 6 years, and the amount that should be carried in the depreciation account at the end of 5 years.

2. A taxicab company has purchased 30 cabs costing $1250 each. The estimated life of a cab is 4 years, and its scrap value is $200. Construct a schedule showing book values and depreciation account if a sinking fund earning 3% is used.

3. A machine costing $9500 must be replaced in 12 years, when its scrap value will be $175. Using the constant percentage method, find the book values at the end of each of the first 4 years and also the amounts in the depreciation account. What percentage of the book value is written off each year?

4. A plant consists of 3 parts, with costs, scrap values, and probable lives as follows:

Part	Cost	Scrap Value	Life
A	$100,000	$20,000	40 years
B	40,000	8,000	10 years
C	12,000	600	15 years

Replacement is to be provided for by a sinking fund earning 3½% annually. How much will be in the fund at the end of 12 years?

5. A building costing $50,000, has a probable life of 30 years and a scrap value of $7000. A sinking fund into which semiannual payments are made has been created, the fund earning interest at the rate $j_{(2)} = 2\%$. What will be the book value of the building at the end of 18 years?

6. If the constant percentage method were used, what would be the book value of the building in Exercise 5?

7. See Exercise 7, list A.

68. Unit cost method.* It should be clearly realized that there is something arbitrary in any method of arriving at a book value for a machine or a building, and constructing a replacement fund. The methods that have been given are all reasonable, but the book values arrived at may not be what could be realized from the sale of the article during the period covered by the schedule, for only the scrap or terminal value has been estimated. Nor is the book value the same as might be put down for tax or other purposes, especially in the case of utilities whose charges are subject to public regulations. In fact, we have a problem that we have not so far considered if we wish to determine what might be called a reasonable value for some article at a date between the time when it was new, and when it should be scrapped. Suppose, for instance, that a man is considering the purchase of a plant equipped with machinery as an alternative to building and outfitting a new one. The equipment of the old plant will be somewhat worn, and some of it may be out of date and fall quite short of the production capacity of newer machines. It is quite evident that he is not at all interested in the book values the present owners give their equipment. The sound procedure for him is to consider values for the equipment such that the production cost per unit of product will be essentially the same as that for new machines. Of course estimates must be made that involve uncertainties, but, accepting this condition, an equation can be set up that gives a solution of the problem at issue.

It is not difficult to give a reasonable figure for the production cost per unit by a machine in terms of its value, its cost of operation, the number of units of product it turns out, and other quantities subject to estimation.

Let

V = value of the machine,
n = number of years of remaining life,

---

* In the remaining sections of the chapter, no illustrative examples are given. The student by this time should be able to use the formulas without such assistance.

E = annual charge for operation and upkeep,
S = scrap value,
i = interest rate earned by a sinking fund,
Y = number of units the machine produces in a year,
U = unit cost per article made.

The amount that should be paid into a sinking fund each year is

$$(V - S)\frac{1}{s_{\overline{n}|}}.$$

Let us also include as an annual charge the interest $i \cdot V$ on the value of the machine. Of course, the interest earned on the sinking fund is usually small, and the part of the original outlay already paid into it is earning such interest, so that by adding $i \cdot V$ as an annual charge we are providing for a fairly reasonable return on invested capital. The total yearly charge on the machine can thus be taken as

$$(V - S)\frac{1}{s_{\overline{n}|}} + iV + E,$$

or, making use of the relation

$$\frac{1}{s_{\overline{n}|}} + i = \frac{1}{a_{\overline{n}|}},$$

we can write the yearly charge in either of the two additional ways:

$$(V - S)\frac{1}{a_{\overline{n}|}} + iS + E, \quad V\frac{1}{a_{\overline{n}|}} - S\frac{1}{s_{\overline{n}|}} + E.$$

Accordingly we have the three following expressions for the unit cost

$$(8) \qquad U = \frac{(V - S)\frac{1}{s_{\overline{n}|}} + iV + E}{Y} = \frac{(V - S)\frac{1}{a_{\overline{n}|}} + iS + E}{Y} = \frac{V\frac{1}{a_{\overline{n}|}} - S\frac{1}{s_{\overline{n}|}} + E}{Y}.$$

In case one is computing U for a machine for which V, S, E, and Y are all known, the first or second form is usually to to be preferred. (Why? Has either expression any advantage over the other?) If three of the four quantities are known, the other can be determined so as to cause the unit cost of the article to have a specified value. In finding the value to give an old machine so that its product will have the same cost per unit as that for a new machine, V will be the unknown, and the last form should be used in writing the appropriate equation.

If it is desired that the capital V invested in the machine should earn at rate i' the unit cost will be

$$(9) \qquad U = \frac{(V - S)\frac{1}{s_{\overline{n}|i}} + i'V + E}{Y}.$$

One can also include in E a proper portion of general overhead cost of a plant or a part of a plant.

Of course it would be possible to make use of (8) for the annual determination of the book value of some article, but reasons for not doing so are quite clear, and the chief value of the formula lies in estimating the actual worth for some used machine, or for comparing the relative prices of two new machines with different productive capacities, operating charges, and so on. The determination of unit cost can also evidently be used in arriving at lower limits at which articles can be sold without suffering loss.

EXERCISES

A

1. A machine costing \$4000 has an estimated life of 10 years, and a scrap value of \$200. Wages for operating it are \$1200 a year, and other operating expenses amount to \$600 a year. If it produces 1200 units a year, what is the cost per unit on a 5% basis?

2. Compared with the machine in Exercise 1, what is the value of a used machine that requires \$1000 a year to operate, \$300 a year for other expenses, produces 800 similar units, and has a probable life of 6 years with a scrap value of \$150, also evaluated on a 5% basis?

3. One machine that can be bought for \$8000 requires \$2000 a year to operate, can produce 1500 units, and has a probable life of 12 years and a scrap value of \$300. A second machine, with a probable life of 18 years, a scrap value of \$400, and a yearly capacity of 2000 units, can be purchased for \$12,000. The second machine is known to be more expensive to operate. What yearly operating cost would make it comparable in value to the first machine on a 4% basis?

4. Suppose that the second machine in Exercise 3 costs \$2200 a year to operate. At what price should it sell in order to be comparable with the first machine, if the other characteristics of the machine are left unchanged?

5. A machine costing \$20,000 is estimated to have a life of 20 years and a scrap value of \$2000. A total annual charge of \$8000 is made against the machine for operation and overhead expenses. How many units must it turn out in a year to compete with a machine whose unit cost is 65.24 cents, if a sinking fund earning $2\frac{1}{2}\%$ is used to recover the wearing value of the machine, but it is desired to earn 5% on the money invested in the machine?

B

1. A machine costing \$12,500 has an estimated life of 15 years and a scrap value of \$800. Wages for operating it are \$3000 a year, and other expenses amount to \$250 a year. If it produces 10,000 units a year, what is the unit cost on a 3% basis?

2. Compared with the machine in Exercise 1, what is the value of a used machine that requires \$2400 a year to operate, \$300 a year for other expenses, produces 8000 units a year, and has a probable life of 8 years with a scrap value of \$150?

3. One machine that can be bought for \$5000 requires \$1800 a year to operate, can produce 3000 units, and has a probable life of 15 years and a scrap value of \$250. A second machine with a probable life of 20 years, a scrap value of \$300, and a yearly capacity of 4000 units can be bought for \$7000. The second machine is more expensive to operate. What yearly operating cost would make it comparable in value with the first machine on a 5% basis?

4. Suppose that the second machine in Exercise 3 costs \$2000 a year to operate. At what price should it sell in order to be comparable with the first machine, if the other characteristics of the machine are left unchanged?

5. A machine costing \$18,000 is estimated to have a life of 15 years and a scrap value of \$1500. Operation and overhead charges against the machine amount to \$6000 a year. How many units must the machine turn out to compete with a machine whose unit cost is 36.48 cents, if a sinking fund earning 2% is used to recover the wearing value of the machine, but it is desired to earn 6% on the money invested in the machine?

69. Composite life of a plant. The various parts of a plant generally have to be renewed at different times. But it is natural to ask whether it is possible to assign some reasonable measure to the life of the plant as a whole, based upon the values and lives of the component parts. It is clear that the basis for a reasonable estimate of the sort desired lies in considering the amount that must be put yearly into a sinking fund to replace the various parts, as well as the total depreciation of the plant.

Suppose that the plant consists of r parts, and that their respective wearing values are

$$W_1, W_2, \cdots, W_r.$$

Let the amounts that must be put annually into a sinking fund because of the different parts be respectively

$$D_1, D_2, \cdots, D_r.$$

Let

$$W = W_1 + W_2 + \cdots + W_r, \quad D = D_1 + D_2 + \cdots + D_r, \tag{10}$$

so that W is the total wearing value of the plant and D is the total annual payment into the sinking fund.

Let n be defined by the equation

$$D \cdot s_{\overline{n|}} = W.$$

Then n is the number of years in which the annual payment D would accumulate to the total wearing value W of the plant. Quite appropriately n is called the *composite life* of the plant.

The value of n is found by computing

$$s_{\overline{n|}} = \frac{W}{D} \tag{11}$$

and taking n from the appropriate table. If the expression for $s_{\overline{n|}}$ is used, it is possible to find n by logarithms. (See Exercise 1, page 61.) As the integer closest to n is all that is desired, the tabular solution without interpolation is sufficient.

The yearly payments $D_1, D_2, \cdots, D_r$ to be made into the sinking fund on account of the different parts of the plant should be found by the sinking fund method described in section 65.

If bonds have been sold to provide in part for the construction of a plant, the term of the bonds should be less than the composite life of the plant by a safe margin. The reason for this is quite apparent.

EXERCISES

A

1. Find the composite life of the plant in Exercise 4, list A, page 116, on a 4% basis.

2. A plant consists of the following parts: (a) building, cost \$200,000, life 40 years, scrap value \$10,000; (b) furniture, cost \$10,000, life 15 years, scrap value \$1000; (c) machinery, cost \$150,000, life 12 years, scrap value \$5000; (d) heating plant, etc , cost \$20,000, life 20 years, scrap value \$2000 Find the composite life of the plant on a 5% basis; on a 6% basis; on a $3\frac{1}{2}$% basis.

3. If the straight line method of depreciation is used, show that the composite life is given by

$$n = \frac{W_1 + W_2 + \cdots + W_r}{\dfrac{W_1}{n_1} + \dfrac{W_2}{n_2} + \cdots + \dfrac{W_r}{n_r}},$$

where $n_1, n_2, \cdots, n_r$ are the lives of the different parts.

4. Find the composite life of the plant in Exercise 2, using the straight line method.

B

1. Find the composite life of a plant in Exercise 4, list B, page 116 on a 3% basis.

2. A plant consists of the following parts· (a) building, cost \$150,000, life 35 years, scrap value \$8000; (b) furniture, cost \$8000, life 12 years, scrap value \$600; (c) machinery, cost \$110,000, life 15 years, scrap value \$8000; (d) heating plant, etc., cost \$15,000, life 18 years, scrap value \$1800. Find the composite life on a 4% basis.

3. Exercise 3, list A.

4. Find the composite life of the plant in Exercise 2 on the straight line basis.

70. Perpetuities and capitalized cost. It is frequently desired to know how much money is needed to provide for some initial building, and for operation and repairs over an indefinite period. This is particularly true for donations to institutions. Many colleges, for instance, have been handicapped by having money given for a building without the necessary funds for operation and repairs. The building then becomes a drain on the general income of the college, and expenditure for instruction may have to be decreased. On this account some institutions will not accept buildings unless they are accompanied by appropriate endowments.

The basic problem is this: How large an endowment X drawing interest at rate i will support an expenditure S every r years?

The interest earned each year is $i \cdot X$, and if the endowment is not to be impaired, that is, if an indefinite number of expenditures is to be provided for, the yearly interest $i \cdot X$ must accumulate to S every r years. That is,*

$$iX \cdot s_{\overline{r}|} = S.$$

Hence

$$X = \frac{S}{i} \cdot \frac{1}{s_{\overline{r}|}}. \tag{12}$$

If the charge in question is an annual one, so that $r = 1$, the result takes a simple form, for

$$s_{\overline{1}|} = \frac{(1+i) - 1}{i} = 1,$$

and

$$X = \frac{S}{i}. \tag{13}$$

This result could have been written down at once.

An expenditure that is to be made at regular intervals over an indefinitely large number of years is called a *perpetuity*.

The sum of the original outlay and the endowments necessary for operation and repairs is called the *capitalized cost* of a building or other establishment. (There are occasions where it might be desired to have the term *capitalized cost* refer to future maintenance and repairs with the omission of the original construction cost. The interpretation should be clear from the context.)

There are cases where the object in question must be actually rebuilt at regular periods. Of course, building costs and materials will vary from time to time, but if the assumption is made that they remain fixed, a simple expression is found for the capitalized cost. Let C = initial cost; then also $S = C$. If we denote the capitalized cost by C_∞, the "infinity" symbol signifying an indefinite number of renewals, we have

$$C_\infty = C + \frac{C}{i} \cdot \frac{1}{s_{\overline{r}|}}.$$

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* It is assumed in the equation as written that the income iX is also invested at rate i. It may not of course be practical to do this if the interest on the endowment is large. On the other hand, if the endowment were paying a low interest, and it were kept low for safety, it might be possible to have the interest from the endowment receive a larger rate. Modification of details are not difficult to make.

Using the relation

$$\frac{1}{s_{\overline{r}|}} = \frac{1}{a_{\overline{r}|}} - i,$$

we obtain

$$(14) \qquad C_\infty = \frac{C}{i} \cdot \frac{1}{a_{\overline{r}|}}.$$

EXERCISES

A

1. What endowment on a 2½% basis is necessary for a building with an estimated yearly charge (heat, light, water, janitor service, minor repairs) of $15,000, and a charge of $20,000 every 10 years for major repairs, renovations, and replacements?

2. One type of building material costs $4 per unit and will last 10 years. A second type will last 15 years. What should be the cost of the second per unit if the two are capitalized on a 5½% basis?

3. A man left $100,000 as endowment for a college building, and the funds were invested at 5%. The trustees decided to use $4000 a year of the income for current operating expenses. How much was available every 15 years for major repairs, furnishings, and equipment, if the unexpended yearly income from the endowment was put in a savings account paying 3%?

4. The sum of $10,000 has been given to a college for a fellowship. The trustees decide to use it as a traveling fellowship with a stipend of approximately $2000 a year. How often can the fellowship be awarded if the funds are invested at 3½%, and what will be the payment the endowment will support?

5. Show that the formula for $a_{\overline{n}|}$ tends toward the limit $1/i$ when n increases indefinitely, and explain why it should do so.

B

1. What endowment on a 4½% basis is necessary for a building with an estimated yearly charge (heat, light, water, janitor service, minor repairs) of $10,000, and a charge of $15,000 every 12 years for major repairs, renovations, and replacements?

2. One type of building material costs $3.50 per unit and will last 12 years. A better type will last 18 years. What should be the cost of the second per unit if the two are capitalized on a 5% basis?

3. A man left $80,000 as endowment for a college building, and the funds were invested at 4%. The trustees decided to use $3500 a year of the income for current operating expenses. How much will be available every 12 years for major repairs, refurnishing, and equipment, if the unexpended yearly income from the endowment is put in a savings account paying 2%?

4. The sum of $25,000 has been given a college for a professorship. The trustees decide to use it for a visiting professorship paying approximately

$5000 a year. How often can the professorship be awarded, and what will it pay if the funds are invested at 3%?

5. Exercise 5, list A.

71. Evaluation of mining properties. The question of depreciation is especially important in the case of natural resources, such as mines or quarries. Here total exhaustion of the property as a revenue producer will take place after a certain interval of time. An engineer by a careful survey may determine how long the property will last, with a certain annual production. Assuming that the annual output can be sold at estimated prices, a more or less accurate figure for annual net income can be obtained. Part of this income must go into a sinking fund to restore the original investment in the property, the remainder being profit. From the estimated data it is then possible to arrive at a figure expressing the value of the property.

Let V = value of the property,
n = its estimated life,
R = estimated annual income after operating charges, etc., have been paid,
i' = investment rate, usually high in such cases,
i = sinking fund rate.

The amount that must be paid into the sinking fund each year is

$$V \cdot \frac{1}{s_{\overline{n}|}} \quad (\text{rate } i),$$

for it is assumed that V is the price that is to be paid for the property and putting it into operating condition. Certain equipment may have a remaining value, but this is not here taken into account.

The required income on investment being $i'V$, we have for sinking fund purposes $R - i'V$.

Hence

$$R - i'V = V \cdot \frac{1}{s_{\overline{n}|i}},$$

so that

$$V = \frac{R}{\dfrac{1}{s_{\overline{n}|i}} + i'}. \tag{15}$$

The problem of estimating for tax purposes properties of the sort here considered is particularly difficult. The term *depletion* is used by the government to denote the diminution of the value of the property due to the using up of the product that comes from it.

EXERCISES

It is estimated that a mine will give a net annual income of $15,000 and will be exhausted in 20 years. What is its value if a sinking fund earning 3% is used, and the investment rate is (a) 8%? (b) 12%?

CHAPTER IX

LIFE ANNUITIES

72. Mortality tables. Edmund Halley (1656–1742), the great friend of Isaac Newton (1642–1727), is known particularly for his eminence as an astronomer. To him also belongs, however, the distinction and the honor of laying the foundation for life annuities and life insurance by constructing the first *mortality table.* Such a table gives the number of persons living at different ages out of an original group of stated size, and is the basis for any consideration of problems involving the contingency of human life. Halley studied the records of births and deaths in the city of Breslau, Silesia, for the five-year period 1687–1691. The figures showed that the population could be regarded as practically *stationary*, not many persons moving away and not many moving in. The table that Halley constructed was published in 1693; by means of it he solved several problems concerning life annuities, but about 70 years elapsed before the formation of a life insurance company on a scientific basis. Since that time *actuarial science** has developed into an extensive subject, and is one of the socially most important studies of the present day. The records of life insurance companies and the modern census give means of constructing mortality tables far more accurate than the early tables.

73. The American Experience and the Standard Annuity Tables. A large number of life insurance companies located in the United States use the American Experience Table of Mortality in connection with their life insurance calculations. This table is given as Table XI in the present book. The

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* The word *actuary*, derived from the Latin *actuarius*, meaning "copyist," was originally used to denote a registrar or clerk in a broad sense. It is now employed in a very exclusive way, and designates a person thoroughly acquainted with the mathematical principles necessary for the scientific calculation of life insurance risks.

American Experience Table was first published in 1868,* though the data on which it was based are not altogether certain. In this table the basic group of persons is one of 100,000 alive at age 10, and the table shows how many of them are alive at age 11, 12, 13, $\cdots$, up to 96, the terminal age for the table.

In actuarial science the symbol l_x is used to denote the number of persons of the original group who are alive at age x. In the American Experience Table x varies from 10 to 96 (the entry $l_{96} = 0$, however, being omitted). We find, for example,

$$l_{20} = 92{,}637, \qquad l_{45} = 74{,}173, \qquad l_{80} = 14{,}474.$$

That is, out of the original group of 100,000 persons alive at age 10, there are 92,637 alive at age 20, and so on.

There are obvious reasons why a mortality table will not continue to represent actual experience over a long period, though it may have been skillfully constructed on good data. Chief among the reasons are improvements in general hygiene and medical science, which have in recent years been very notable. For example, deaths from typhoid fever have been greatly reduced, and some progress has been made against tuberculosis. Further success in combating tuberculosis and an effective control of cancer would be reflected in mortality experience.

The 80-year old American Experience Table cannot be expected accurately to represent present-day experience. But its use in connection with life insurance tends toward conservatism, and gives added safety to policies. As companies have fewer deaths at earlier ages than their calculations allow for, profit comes to them, to be generally passed on to policyholders as dividends, to be explained later.

The case, however, is different when we come to questions of annuities. Here payments are to be made to individuals during years that they live, and with mortality lower than a table predicts for earlier ages, it would be possible for a company to suffer heavy losses on annuities.

For annuities, companies very generally employ a table that more correctly represents present mortality than does the American Experience Table. In Table XIII there is given part of the 1937 Standard Annuity Mortality Table,* which is often used at present. Since the mortality of men and women differs considerably, different figures are used for the same age, but by employing a "set-back" principle, it is possible to use a single table. The complete table extends to age 110 for men and to age 115 for women.

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* It was constructed by Sheppard Homans and was published in a schedule in an act of the legislature of New York on May 6, 1868.

Example 1. Of 10,000 persons of age 30, how many will be alive at age 50, according to the American Experience Table?

$$10{,}000\,\frac{l_{50}}{l_{30}} = \frac{698{,}040{,}000}{85{,}441} = 8170.$$

Example 2. Of 10,000 men of age 30, how many will be alive at age 50, according to the Standard Annuity Table?

$$10{,}000\,\frac{l_{50}}{l_{30}} = \frac{8{,}806{,}970{,}000}{965{,}939} = 9118.$$

EXERCISES

1. Of 10,000 women of age 30, how many will be alive at age 50, according to the Standard Annuity Table?

2. Of 125 persons of age 42, how many will be alive at age 62, according to the American Experience Table?

3. Explain how the following numbers are obtained from the Standard Annuity Table for men as a basis for comparison with the American Experience Table.

x	l_x	x	l_x
20	92,637	25	91,988
21	92,514	26	91,844
22	92,388	27	91,694
23	92,259	28	91,536
24	92,126	29	91,369

Obtain the number dying at different ages and compare with the American Experience Table.

74. Definitions. A *life annuity* is an annuity to be paid to a person throughout all or a portion of his life. If it is paid throughout all the years subsequent to the date of its purchase it is a *whole life annuity;* if payments are not to begin until after the passing of a certain number of years, it is called a *deferred life annuity;* if payments are to take place for only a specified period, and then cease, even though the person is alive, it is a *temporary annuity.*

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* The table is taken from an article by Frank D. Kineke, "A New Annuity Mortality Table," *Transactions Actuarial Society of America*, Vol. 39 (1938), pp. 8–23. Commutations columns with rates other than 3% are given in Vol. 40.

Since the number of payments to be made on a life annuity cannot be foreseen, it is called a *contingent annuity.* In contrast, annuities such as those considered in the earlier chapters, in which a definite number of payments is stipulated at the outset, are called *annuities certain.*

We have two reasons for considering annuities, and they lead to the use of different mortality tables. In the first place the theory of annuities is, as will be shown in the next chapter, associated closely with that of life insurance, though at first the two ideas seem quite unrelated. In such a connection the mortality table used in connection with life insurance is to be employed. As has been stated, the table generally used is the American Experience Table, which adds safety to an insurance policy on account of the high tabular mortality at the younger ages.

In the second place, annuities are actually sold by insurance companies, and there is a tendency for this business to increase. When considered from this standpoint, a mortality table such as the Standard Annuity Table should be used. The terms of the annuity are embodied in a contract called the *policy;* the person who is to receive the payments is called the *annuitant;** the sum (or sums) paid for the annuity is called the *premium* (or premiums). In this chapter we consider the *net premiums* (with the exception of the so-called retirement annuities). In practice these would be increased by certain *loadings* to provide for the agent's commission and other expenses, in order to obtain the *gross* or *office premium,* at which the annuity would actually be sold. The manner of determining such loadings is not the same for all companies.

In addition to selecting a mortality table, it is necessary to assume some interest rate in computing premiums on annuities and insurance. The money which policyholders pay is kept invested by the companies, and the interest earned affects the price the purchaser must pay for a contract. The investments of insurance companies should, first of all, be sound. An interest rate of 3½% was commonly used in calculating insurance premiums until about 1935. Decrease in the income received from investments then compelled companies to lower their rates to 3%, or even to 2½%. Most of the present outstanding contracts were written with the higher rates, and they therefore remain important. In this book tables on a 3% basis are given for both the American Experience Table and the Standard Annuity Mortality Table.

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* In the case of a deferred annuity, there is some objection to speaking of the person who is to receive the payments as the *annuitant,* during the period prior to the receipt of payments. He is sometimes called the *nominee.*

The words *policy* and *premiums* have at times been reserved for life insurance, and the words *contract* and *consideration* used for annuities.

75. Pure endowment. We base the subject of annuities on a simpler idea, that of a *pure endowment.* If a person of age x is to receive the single payment of 1 if he is alive in n years, there is on his life a pure endowment. Let ${}_nE_x$ be the net charge for such a contract. The value desired is easily obtained by assuming that the company is selling such contracts to the l_x persons of age x. In n years the number of persons living, each of whom will be paid 1, will be l_{x+n}; thus the company must pay the amount l_{x+n} in n years. In order to provide this, it must "now" have the amount

$$v^n l_{x+n}, \qquad v = \frac{1}{1+i}.$$

The company must collect this amount from the entire group of l_x persons in order to give them each the contract. The cost per person is then

$$(1) \qquad {}_nE_x = \frac{v^n \cdot l_{x+n}}{l_x}.$$

As it stands, the expression for ${}_nE_x$, though simple in appearance, has a serious objection. Different values of x and n will give the same sum $x + n$. In such cases the same number l_{x+n} will occur, but it will be multiplied by different powers of v. For instance, we have

$${}_{20}E_{20} = \frac{v^{20}l_{40}}{l_{20}}, \qquad {}_{15}E_{25} = \frac{v^{15}l_{40}}{l_{25}}, \qquad {}_{10}E_{30} = \frac{v^{10}l_{40}}{l_{30}}.$$

In all of these expressions l_{40} occurs, but multiplied by different factors. It thus appears that although ${}_nE_x$ can be computed by using the mortality table and a table of powers of v, an actual calculation will be needed in forming the numerator, or an extensive table will be necessary to take account of all the combinations that may be required.

By a very simple device the objection described can be overcome, and the student is urged to appreciate the significance of the transformation made.

Multiply the numerator and denominator in (1) by v^x; we then have

$$(2) \qquad {}_nE_x = \frac{v^{x+n} \cdot l_{x+n}}{v^x l_x}.$$

Let us put

$$(3) \qquad D_r = v^r \cdot l_r.$$

The number D_r is a definite quantity for each year of age, and can be tabulated by using the numbers in the mortality table and the powers of v from the table for that quantity. Using the American Experience Table and letting $i = 3\%$ we have, for example,

$$D_{25} = v^{25} l_{25} = (.477\,6056)(89{,}032) = 42{,}522.18.$$

The value of D_r* is given in Table XII for the American Experience Table, and in Table XIV for the Standard Annuity Table, the interest rate being 3% in each case. The quantity is called a *commutation symbol.*

In terms of the symbol D we have

$$(4) \qquad {}_nE_x = \frac{D_{x+n}}{D_x}.$$

When it is desired to have the American Experience Table used, an example or exercise will refer to a *person* (that table making no distinction between men and women). If it is desired to have the Standard Annuity Table used, the reference will be to a *man* or a *woman.*

Example. What is the net cost of an endowment of $1000 due in 25 years on the life of a person 30 years of age?

The symbolic answer is

$$\$1000\,{}_{25}E_{30} = \$1000 \frac{D_{55}}{D_{30}}.$$

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* In the tables the heading is D_x. No significance is to be attached to the particular letter used as a subscript.

From Table XII we find $D_{30} = 35{,}200.6$, $D_{55} = 12{,}703.9$; hence the cost is

$$\frac{\$12{,}703{,}900}{35{,}200.6} = \$360.90.$$

On the other hand, the amount that must be deposited in a bank paying 3% in order that $1000 can be withdrawn in 25 years is

$$\$1000v^{25}(3\%) = \$477.61.$$

The lower price of the endowment arises from the fact that the

$$l_{30} - l_{55} = 85{,}441 - 64{,}563 = 20{,}878$$

persons who die, according to the American Experience Table, contribute to the payments to those who survive.

EXERCISES

1. Find D_{40} for men and women. Check with Table XIV and explain.

2. Find the values of endowments of $2000 due in 20 years on the life of a man and the life of a woman, respectively, both aged 45.

3. Find the values of endowments of $5000 due in 40 years for both a man and a woman aged 25.

4. Find the premium for an endowment of $5000 due in 15 years on the life of a person aged 35.

5. Find the amounts of the payments in Exercise 2 if accumulated at 3% for 20 years.

76. Life annuities. A life annuity pays 1 at the end of every year a person is alive, the first payment being received one year from the date of purchase of the contract. Let a_x be the net premium for a person aged x. The contract can be considered as a group of pure endowments, due in 1, 2, 3, etc., years.

Hence

$$a_x = {}_1E_x + {}_2E_x + {}_3E_x + \cdots \text{ to table limit.}$$

Using (4) we have

$$a_x = \frac{D_{x+1} + D_{x+2} + D_{x+3} + \cdots \text{ to table limit}}{D_x}.$$

In general, let *

$$(5) \qquad N_r = D_r + D_{r+1} + D_{r+2} + \cdots \text{ to table limit.}$$

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* The letter N is called an "open bar" letter. There is an unfortunate lack of uniformity of practice. The notation above is that generally used in America. The International Congress of Actuaries has adopted the symbol

$$N_r = D_{r+1} + D_{r+2} + \cdots.$$

It is seen at once that the quantity D_r can be easily tabulated. We have in fact

$$N_r = D_r + (D_{r+1} + D_{r+2} + \cdots);$$

that is

$$N_r = D_r + N_{r+1}. \tag{6}$$

Further, taking the case of the American Experience Table,†

$$N_{95} = D_{95}. \tag{7}$$

Using (6) and (7), the column for N_r in Table XII is formed by starting at the end and working back.

We now have

$$a_x = \frac{N_{x+1}}{D_x}. \tag{8}$$

and a very important and seemingly difficult problem is easily solved by means of an appropriate table.

Example. What is the net premium for a life annuity of \$500 a year on the life of a person aged 30?

In symbols the premium is

$$\$500 a_{30} = \$500 \frac{N_{31}}{D_{30}}.$$

From the table, $N_{31} = 707{,}283$, $D_{30} = 35{,}200.6$. Hence the net premium is

$$\frac{\$353{,}641{,}500}{35{,}200.6} = \$10{,}046.$$

It should be well understood that even though the person dies during the first year, he leaves nothing to his family or his estate. In one sense he has "beneficiaries," but they are the other persons of age 30 who buy annuities from the same company he bought of, and who live a long time.

EXERCISES

1. Find N_{50} (American Experience), given that $D_{50} = 15{,}922.8$, $N_{51} = 227{,}233$.

2. Find the net premium for a life annuity of \$1000 a year on the life of a man aged 50, a woman aged 50.

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† In general,

$$N_{\omega-1} = D_{\omega-1},$$

where ω is the table limit.

3. Compare the first result in Exercise 2 with the cost of an annuity certain of $1000 a year for 25 years, on a 3% basis.

4. A woman aged 38 has been left $10,000. What life annuity could she purchase with it?

5. Show that the net single premium for an annuity of 1 per year on the life of a person aged x, payments to be made throughout the life of the annuitant and for 2 years after his death, the last 2 payments to a beneficiary, is

$$\frac{vl_x + v^2l_x + v^3l_{x+1} + v^4l_{x+2} + \cdots}{l_x}.$$

Show that the result can be written

$$v + (1 + a_x)v^2.$$

6. Similar to Exercise 5, 3 payments being made after the death of the annuitant.

7. Should a company require a medical examination before selling an annuity contract?

8. The annual dues in a university alumni association are $3. How much should the association charge a person aged 21 for life membership? Should the money collected from a life membership all be used the year it is collected? How should the life membership fund be handled?

77. Deferred annuities. By a life annuity deferred n years is meant one in which the first payment is made in $n + 1$ years, payments then continuing throughout the life of the annuitant. Let ${}_{n|}a_x$ be the net single premium of an annuity of 1 per year, deferred n years, on the life of a person aged x. It is seen that

$${}_{n|}a_x = {}_{n+1}E_x + {}_{n+2}E_x + \cdots \text{ to table limit},$$

or, using (4)

$${}_{n|}a_x = \frac{D_{x+n+1} + D_{x+n+2} + \cdots \text{ to table limit}}{D_x},$$

and thus by (5)

$$(9) \qquad {}_{n|}\boldsymbol{a}_x = \frac{\mathbf{N}_{x+n+1}}{\boldsymbol{D}_x}.$$

It is to be observed that the subscript to N is the age at which the first payment is received.

Example. What is the net single premium for an annuity of $1000 a year deferred 20 years, on the life of a person aged 30?

The required premium is

$$\$1000_{20|}a_{30} = \$1000\frac{N_{51}}{D_{30}} = \$6455.37.$$

EXERCISES

1. Out of 1000 persons of age 30 who purchase a life annuity deferred 20 years, how many, according to the American Experience table, do not live to receive any payments? How many receive just one payment? How many just 2 payments?

2. What is the net single premium for an annuity of \$500 a year deferred 15 years, on the life of a man aged 45?

3. Compare the result in Exercise 2 with the cost, on a 3% basis, of an annuity certain of \$500, deferred 15 years, and consisting of 25 payments.

4. A woman aged 30 has received \$5000 from a will. How large a life annuity can she purchase, the first payment to be received at age 50? What annuity would she receive if she deposited the money in a bank paying 3%, and at age 49 bought the annuity?

5. Show that the net premium for a life annuity of 1 purchased at age x, with the first 5 payments *certain*, that is, paid in case of the death of the annuitant, is

$$a_{\overline{5|}} + {}_{5|}a_x.$$

6. What is the net premium for a life annuity of \$500, with the first 5 payments certain, on the life of a person aged 30, $a_{\overline{5|}}$ being evaluated on a 3% basis? Compare with the example on page 134.

7. A man 43 years old has been left \$10,000. How large a life annuity with 10 certain payments can he purchase if the annuity certain that is involved is evaluated on a 3% basis?

8. Suppose that the amount n is paid by a person aged x for a life annuity of 1 a year, with n payments certain. Show that n is to be found from the equation

$$a_{\overline{n|}} + {}_{n|}a_x = n.$$

Such an annuity is called a *refund annuity*, since it guarantees the repayment at least of the purchase price.

For a given value of x it is not especially difficult to find by trial the integral value for n that comes nearest satisfying the equation.

9. What is the cost of a refund annuity of 1 a year for a person aged 35, and how many payments are guaranteed? Compare the cost with that of an ordinary life annuity.

SOLUTION: It is necessary to find n such that

$$a_{\overline{n|}} + {}_{n|}a_{35} = n.$$

Obtain the following results:

n	$a_{\overline{n|}}$	${}_{n|}a_{35}$	$a_{\overline{n|}} + {}_{|n}a_{35}$
20	14.8775	5.4142	20.2917
21	15.4150	4.9979	20.4129

Hence the cost is 21, and 21 payments of 1 each are guaranteed.*

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* It is recalled that in practice the net premium must be increased, but it is not implied that the correct loading would make the gross cost of the annuity considered exactly 21.

The cost of an ordinary life annuity is $a_{35} = 18.9174$.

10. A person aged 35 has decided to invest \$5000 in an annuity. Compare the yearly incomes from the refund annuity and the ordinary life annuity that could be purchased.

11. Find the yearly income from a refund annuity purchased for \$4000 by a man aged 50. Compare the income with that from an ordinary annuity.

12. What refund annuity could the woman in Exercise 4, page 136, have purchased?

78. Temporary annuities. Since the payments from a temporary annuity cease after a certain period, even though the annuitant may be alive, it is clear that such an annuity is not a very attractive one to buy. The idea, however, has importance, because it is used as the basis for obtaining results of wide application.

Let $a_{x\overline{n|}}$ be the net premium for a temporary annuity of 1 for n years on the life of a person aged x.* It is seen that if a person purchases a temporary annuity for n years and also an annuity deferred n years, he will in effect have purchased a life annuity. Hence

$$a_x = {}_{n|}a_x + a_{x\overline{n|}},$$

so that

$$a_{x\overline{n|}} = a_x - {}_{n|}a_x.$$

Using (8) and (9) we have

$$a_{x\overline{n|}} = \frac{N_{x+1} - N_{x+n+1}}{D_x}. \tag{10}$$

79. Auxiliary theorems. There would be little occasion for selling a *life annuity due*, that is, one in which the first payment is made at once, but the conception is useful. Representing the net single premium for such an annuity by a_x, we have

$$\mathrm{a}_x = 1 + a_x = 1 + \frac{N_{x+1}}{D_x} = \frac{N_x}{D_x}, \tag{11}$$

equation (6) having been used to obtain the final result.

From this it is seen that for the single premium 1 an annuity with yearly payments to the amount of

$$\frac{1}{1 + a_x} = \frac{D_x}{N_x}$$

---

* The premium is also represented by $|_n a_x$. When the form in the text is used, a colon is sometimes inserted, thus $a_{x:\overline{n|}}$.

can be purchased on the life of a person aged x, the first payment being received at once. It follows that, if an insurance company will sell a person aged x a certain contract for a cash payment of 1, it should also sell him the same contract for yearly premiums (to be paid as long as he lives) of D_x/N_x, the first premium to be paid at once.*

We thus arrive at the following theorem:

THEOREM 1. To find the yearly premium to continue throughout the life of a person aged x, first payment due at once, multiply the net single premium for the contract in question by $D_x/N_x (= 1/(1 + a_x) = 1/\text{a}_x)$.

Similarly, the net single premium for a temporary annuity of 1, consisting of n payments, the first payment to be made at once, is

$$\text{a}_{x\,\overline{n|}} = 1 + a_{x\,\overline{n-1|}}.$$

Therefore, a payment of 1 will buy an annuity of n payments on the life of a person aged x, each payment being in amount

$$\frac{1}{\text{a}_{x\,\overline{n|}}} = \frac{1}{1 + a_{x\,\overline{n-1|}}} = \frac{D_x}{N_x - N_{x+n}},$$

the first payment being made at once. It follows, conversely, that if a company will sell a certain contract for the single premium of 1 to a person aged x, it should sell the same contract for n payments (if the person lives to make them all), each of the amount given above, the first payment being made at once. Therefore:

THEOREM 2. To find the yearly premium of a series of at most n payments to be made by a person aged x, multiply the net single premium for the contract in question by

$$\frac{D_x}{N_x - N_{x+n}},$$

the first premium payment being made at once.

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* The exaggerated mortality at early ages given in the American Experience Table tends to lower the value of a_x. This increases the reciprocal of $1 + a_x$, that is, the yearly payment is greater than it would be if a table with lower mortality at earlier years were used. The chief use of the theorems above will be to obtain annual premiums for life insurance policies. The net premiums in such cases are conservatively high, owing to the use of the American Experience Table; converting to yearly premiums tends to increase the safety of the contracts.

80. Annual payment for a deferred annuity. A person may not be able to purchase a deferred annuity by a single premium payment. He may wish to make yearly payments. As he would plan to receive the annuity payments after he has retired, he might well plan his premium payments to continue up to one year before he wishes to receive the first payment from the annuity. Suppose that the first payment from the annuity is to be received in n years. The annuity, being deferred $n - 1$ years, has a net single premium

$$_{n-1|}a_x = \frac{N_{x+n}}{D_x}.$$

If yearly premiums are to begin at once and continue until one year before the first payment from the annuity is received, there will be n premium payments. It follows from Theorem 2 of the last section that the yearly premium is

$$\frac{N_{x+n}}{N_x - N_{x+n}}.$$

Example. A man 45 years old desires an annuity of \$2000 a year, the first payment to be received when he is 65 years old. What will the net annual premium be?

$$\$2000\frac{N_{65}}{N_{45} - N_{65}} = \frac{\$2,317,316,000}{3,398,367} = \$681.89.$$

EXERCISES

1. Find the annual premium to be paid beginning at age 33 in order that a woman can draw a life annuity of \$1500 a year beginning at age 60.

2. A man began at age 35 to pay \$100 yearly for an annuity to begin at age 60. What was the yearly income from the annuity?

3. Prove Theorem 1, page 138, as follows: Suppose that each of the l_x persons owes 1 to the company. Let Y be the equivalent yearly payment, the first payment to be made at once. Show that

$$Y(l_x + vl_{x+1} + v^2l_{x+2} + \cdots) = l_x,$$

and therefore that

$$Y = \frac{1}{1 + a_x}.$$

4. Prove Theorem 2, page 138, in a similar manner.

5. Obtain the expression for the yearly premium for a deferred annuity as follows: Let Y be the premium. The accumulated amount of the payments made by the original group of l_x persons when the nth payment has been made is

$$Y[l_x(1 + i)^{n-1} + l_{x+1}(1 + i)^{n-2} + \cdots + l_{x+n-1}].$$

Show that this can be written

$$\frac{Y}{v^{x+n-1}}\,[N_x - N_{x+n}].$$

This must support a life annuity for l_{x+n-1} persons of age $x + n - 1$, for which purpose the amount

$$l_{x+n-1}\cdot a_{x+n-1}$$

is required. Equate the two expressions and solve for Y.

6. Payments from annuities are generally received more often than once a year; frequently they are received monthly. The precise computation of premiums in such cases would involve assumptions as to deaths throughout a year, and would lead to complicated formulas. The procedure usually followed is simple. We shall consider monthly payments. Since the net premium for an annuity at age x, with first payment due in a year, is a_x, while the premium is $1 + a_x$ if the first payment is made at once, it is natural to adopt the following scheme of payments:

First payment in	Premium
11 months	$a_x + \frac{1}{12}$,
10 months	$a_x + \frac{2}{12}$,
9 months	$a_x + \frac{3}{12}$,
. . . .	. . . .
2 months	$a_x + \frac{10}{12}$,
1 month	$a_x + \frac{11}{12}$.

All the annuities above pay 1 a year. Consider an annuity purchased at age x that is to pay 1 monthly, the first payment in one month. Show that the net premium is

$$\left(a_x + \frac{11}{12}\right) + \left(a_x + \frac{10}{12}\right) + \cdots + \left(a_x + \frac{1}{12}\right) + a_x$$
$$= 12a_x + \frac{11}{2} = 12\left(a_x + \frac{11}{24}\right)$$

If the first payment is to be made at once, show that the net premium is

$$12a_x + \frac{11}{2} + 1 = 12(1 + a_x) - \frac{11}{2} = 12\left(a_x - \frac{11}{24}\right)\cdot$$

7. Suppose that all the l_x persons of age x and their survivors each contribute annually the amount 1 to a fund, invested at rate i, that is to be divided equally among the l_{x+n} survivors who attain age $x + n$. Show that each survivor will receive

$$_nu_x = \frac{l_x(1+i)^n + l_{x+1}(1+i)^{n-1} + \cdots + l_{x+n-1}(1+i)}{l_{x+n}}$$
$$= \frac{D_x + D_{x+1} + \cdots + D_{x+n-1}}{D_{x+n}} = \frac{N_x - N_{x+n}}{D_{x+n}}.$$

8. Show by direct reasoning that the annual premium beginning at age x, for an annuity of 1 per year, beginning at age $x + n$, is given by

$$\frac{\mathfrak{a}_{x+n}}{{}_nu_x} = \frac{1 + a_{x+n}}{{}_nu_x}.$$

81. Annuities with death benefits. *Pure* annuities pay nothing to the dependents of the annuitants who die before any payments have been received. Thus if a man is buying by yearly premiums a deferred annuity to begin in 20 years and dies after he has made 18 payments, nothing is paid to his family. There is no lack of equity in the contract; it is perfectly just on the basis on which it is devised. Many persons, however, desire annuities that will provide something for their dependents, in case they do not survive to receive many or any payments. The refund annuity considered in Exercises 8–12, pages 136–137, is one type of such a contract.

A class of so-called *retirement annuities* is coming into considerable favor, such contracts having death benefit provisions that are payable if the annuitant dies before he begins to receive the annuity payments. In such a contract the company acts merely as a bank during the period of the payments by the annuitant, and in case of his death it pays a certain percentage, frequently 87½%, of his payments accumulated with interest from 2½% to 3½%, to a beneficiary named in the policy. A like amount will be paid on the surrender of the contract, with possible reductions during the early years.

Let the person start making payments of 1 a year at age x with the expectation that he will begin to receive annuity payments at age $x + n$. If he dies between ages $x + r - 1$ and $x + r$, $r < n$, the death benefit is

$$k(s_{\overline{r+1}|} - 1),$$

where k is the percentage in question. If the annuitant survives to age $x + n$, the cash value at that time is used to purchase a life annuity. Since the first payment is received at age $x + n$, the yearly payment received is

$$\frac{k(s_{\overline{n+1}|} - 1)}{\mathfrak{a}_{x+n}}.$$

We shall assume that $k = 0.875$, that $s_{\overline{n+1}|}$ is calculated with 3%, and that the Standard Annuity Table with 3% is used to obtain a_{x+n}. These assumptions do not represent universal practice, but they illustrate a type of annuity a person is likely to have offered to him. No death benefit will be paid to the annuitant's family under such contract, any more than under a pure annuity contract, if death takes place after the annuitant has started to receive payments.[*] In a contract such as described, the reduction factor k serves to provide for the necessary expenses of the company.[†]

Example. A man began to pay \$100 a year at age 45 toward a retirement annuity, the first payment to be received at age 65. What will be the annual payment from the annuity?

We have

$$\$100(.875)(s_{\overline{21}|} - 1) = \$87\ 50 \times 27.67649 = \$2421.69.$$

$$a_{65} = \frac{N_{65}}{D_{65}} = \frac{1{,}158{,}658}{100{,}291} = 11.55296.$$

Hence the yearly payment from the annuity is

$$\frac{\$2421.69}{11.55296} = \$209.62.$$

EXERCISES

1. A man began at age 45 to pay \$100 annually for a pure annuity, the first payment to be received at age 65. Show by the example, page 139, that the annual payment from the annuity was

$$\frac{\$100}{.34094} = \$293.31.$$

If he had taken a retirement annuity instead of a pure annuity, how much would be sacrificed for the death benefit provision?[‡]

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[*] A retirement annuity contract is very likely to give the option of an annuity with 5, 10, or some other number of certain payments, and also the option of a refund annuity. See Exercises 5–12, pages 136–137.

[†] Many companies pay dividends on the policy during the period the annuitant is making payments. These may be used to reduce the annual premium, or they may be left with the company, drawing interest, and increasing the annuity which will ultimately be received. Thus the reduction that is caused by the factor k will be largely, if not entirely, counteracted. A company that does not provide for dividends may be expected to use a larger value of k and a larger interest rate than a company that allows dividends.

[‡] It is to be recalled that the premium in the example on page 139 was a net premium, not a gross premium. Had the gross premium been used, the income from the pure annuity would be lowered.

2. A man began at age 35 to pay \$100 yearly for a retirement annuity to begin at age 60. What will be the yearly income from the annuity? Compare with the answer in Exercise 2, page 139.

3. In Exercise 2, what will be the death benefit if the man dies between ages 52 and 53?

4. Suppose that the annuity in the example above is to be paid monthly, the first payment to be received at age 65. Show by Exercise 6, page 140, that the monthly payment is

$$\frac{\$2421.69}{12(11.55296 - .45833)} = \$18\ 19.$$

What will be the monthly payment if the first payment is received at age 65 years, one month?

5. If the annuity in Exercise 2 is to be paid monthly, first payment at age 60, find the monthly payment.

6. If the annuity in Exercise 2 is to be paid monthly, first payment at age 60 years, one month, find the monthly payment.

82. Application of probabilities. The student familiar with the theory of probability should study the questions of life contingencies from that standpoint. Though the basic questions of life annuities and life insurance can be considered independently of the theory of probability, that theory gives the required basis for more difficult questions, such as those involving two lives.

Let ${}_np_x$ be the probability that a person aged x will live n years; then since l_x and l_{x+n} give, respectively, the number of persons alive at ages x and $x + n$, we have by definition

$${}_np_x = \frac{l_{x+n}}{l_x}.$$

If $n = 1$, we obtain the probability that a person will be alive in a year. This quantity, denoted in brief by p_x, is given for the American Experience Table, Table XI.

The probability that a person of age x dies within a year is represented by q_x. We have

$$q_x = \frac{d_x}{l_x} = \frac{l_x - l_{x+1}}{l_x} = 1 - p_x.$$

That $q_x = 1 - p_x$ is known from the general principles of probability. The value of q_x is given in the table.

We recall that if the probability that a person will receive some award is multiplied by the value of the award, we obtain what is called the person's *mathematical expectation*. It is seen that the value of the pure endowment ${}_nE_x$ is merely the present value of the expectation.

The following theorems in the theory of probability are recalled.

THEOREM 1. The probability that both of two independent events will happen is the product of the probabilities of the happenings of the separate events.

THEOREM 2. The probability that one of two mutually exclusive events will happen is the sum of the probabilities of the separate events.

By means of these theorems, questions concerning survivorship of two persons can be answered, and some such questions are considered in the exercises below. Symbols used other than those already given, are as follows:

$|{}_nq_x$ = probability that a person of age x will die within n years.

${}_{n-1}|q_x$ = probability that a person of age x will die at age $x + n - 1$, that is, in the nth year from the present.

${}_np_{xy} = {}_np_x \cdot {}_np_y$ = probability that a person aged x and a person aged y will both live n years.

$|{}_nq_{\overline{xy}}$ = probability that a person aged x and a person aged y will both die within n years.

${}_np_{\overline{xy}}$ = probability that at least one of two persons, aged x and y respectively, will survive n years.

EXERCISES

1. What is the probability that a man 42 years old (a) will be living in 20 years? (b) will die in his 60th year?

2. A man is 35 years old and his wife is 32. What is the probability that the man will die at age 50 and his wife will survive him?

3. A graduating class consists of 40 persons aged 20, 210 persons aged 21, and 25 persons aged 22. How many will probably be alive on the 25th anniversary of graduation?

4. Prove the following relationships:

$$(1)\qquad |{}_nq_x = 1 - {}_np_x.$$

$$(2)\qquad {}_{n-1}|q_x = \frac{d_{x+n-1}}{l_x} = {}_{n-1}p_x - {}_np_x.$$

$$(3)\qquad |{}_nq_{\overline{xy}} = (1 - {}_np_x)(1 - {}_np_y) = 1 - {}_np_x - {}_np_y + {}_np_{xy}.$$

$$(4)\qquad {}_np_{\overline{xy}} = 1 - |{}_nq_{\overline{xy}} = {}_np_x + {}_np_y - {}_np_{xy}.$$

5. Prove that the probability that just one of two persons, aged x and y respectively, shall live n years is

$$|_{n}q_x \cdot {}_{n}p_y + |_{n}q_y \cdot {}_{n}p_x = {}_{n}p_x + {}_{n}p_y - 2\,{}_{n}p_{xy}.$$

6. Two brothers are aged 40 and 30 respectively. What is the probability (a) that the elder will die in his 50th year and the younger will survive him? (b) that at least one of the two will be alive in 25 years?

CHAPTER X

LIFE INSURANCE

83. Definitions. A life annuity assures an income to the annuitant throughout his remaining years. *Life insurance* on the other hand has as its chief purpose the protection of the dependents of a person from the financial loss that they will suffer in the event of his death. The person on whose life the policy is written is called the *insured*. The individual to whom the payment, called the *benefit*, will be made upon the insured's death is called the *beneficiary*. A policy may be taken out payable to the insured's estate, or in favor of a business which depends for its success upon his activities.* Life insurance is also called *life assurance*, on account of the fact that it creates a condition of confidence or assurance on the part of the insured.

Insurance policies are of different types, varying with the sort of protection that is desired and the manner in which the consideration for the insurance, that is, the *premiums*, are paid. Policies with new provisions are arranged from time to time in order more fully to meet requirements of different persons, and present-day policies have a number of options, especially with regard to the manner of payment of benefits. Insurance companies pay claims immediately upon receipt of legal proof of the death of the insured. In this chapter premiums are computed on the assumption that the claim is paid at the end of the policy year; on the average, then, payment would be made six months after death instead of immediately afterward. The premiums obtained here are the basic or the *net premiums*, which must be increased by certain *loadings* in order to obtain the *gross premiums* for which companies actually sell their policies.

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* For obvious reasons one person cannot have a policy written upon the life of another person unless there is a real and evident *insurable interest*.

84. Whole life policy. If 1 is to be paid at the end of the year in which the insured dies, it is said that there is a *whole life insurance* of 1 on his life. The symbol A_x is used for the net single premium for such a policy on the life of a person aged x.

It may seem at first that there is no close connection between a life annuity and a life insurance; there is a relation, however, and A_x is expressible in terms of a_x in a simple manner. To obtain the relation we begin by considering an annuity that pays 1 at the end of every year a person *enters*, instead of at the end of every year he *survives*. The cost of an annuity of 1 to be paid at the *beginning* of every year the annuitant enters is merely an annuity due, and is

$$1 + a_x,$$

and thus the cost if the payment is made at the *end* of every year he enters is

$$v(1 + a_x),$$

for the same number of payments would be made as in the former case, but each payment would be deferred a year.

Next suppose that a person has a contract with a company by which the company agrees to pay 1 at the end of every year the person *enters*, and the person pays the company 1 at the end of every year he *survives*. At the end of every year that the person survives there is a mere exchange of 1 between the person and the company, so that such payments may be ignored. There will in fact be a single effective payment made by the company at the end of the year in which the person dies. That is, the contract described is merely a whole life insurance. For the series of payments the company makes it would charge, as shown above, $v(1 + a_x)$, while for the sum a_x it would provide the person with the money to make his payments. Thus the single effective payment should be furnished for the amount

$$v(1 + a_x) - a_x.$$

Therefore

$$\mathbf{A_x = v(1 + a_x) - a_x.} \tag{1}$$

The derivation just given is very important in calling upon a fundamental understanding of the different conceptions.

But the direct deduction of an expression for A_x is simple and equally important.

Let

$$(2) \qquad d_x = l_x - l_{x+1},$$

so that d_x represents the number of persons who die at age x. A column giving this number for different ages appears in the mortality table. Now, of the l_x persons taking out a whole life insurance, the number of claims that must be paid at the end of 1, 2, 3, $\cdots$, years is respectively

$$d_x, \quad d_{x+1}, \quad d_{x+2}, \quad \cdots.$$

The insurance being for 1, the amount that must be put at interest initially to provide for the different payments is therefore

$$vd_x + v^2d_{x+1} + v^3d_{x+2} + \cdots \text{ to table limit.}$$

This sum is to be obtained by equal payments from all of the l_x persons. Hence the amount each must pay is

$$(3) \qquad A_x = \frac{vd_x + v^2d_{x+1} + v^3d_{x+2} + \cdots \text{ to table limit}}{l_x}.$$

Multiply numerator and denominator in (3) by v^x and put

$$(4) \qquad C_r = v^{r+1}\cdot d_r.$$

Then

$$(5) \qquad A_x = \frac{C_x + C_{x+1} + C_{x+2} + \cdots \text{ to table limit}}{D_x},$$

where D_x has the meaning that it has in Chapter IX. Let

$$(6) \qquad M_x = C_x + C_{x+1} + C_{x+2} + \cdots \text{ to table limit.}$$

The quantity M_x is a commutation symbol, whose value for each age is given in Table XII.*

We now have

$$(7) \qquad A_x = \frac{M_x}{D_x}.$$

Example 1. Find the net single premium for a whole life insurance of \$1000 on the life of a person aged 30.

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* Throughout this chapter, reference is to the American Experience Table.

In symbols the premium is

$$\$1000A_{30} = \$1000\frac{M_{30}}{D_{30}}.$$

From the table, $M_{30} = 13{,}574.8$, $D_{30} = 35{,}200\ 6$, so that the premium is

$$\frac{\$13{,}574{,}800}{35{,}200.6} = \$385.64.$$

Most persons do not desire, or cannot afford, to purchase insurance by a single payment, but prefer to pay for it either yearly throughout life, or for a limited period.

Suppose the payments are to continue throughout life. The symbol P_x is used for the *net annual premium* on a whole life insurance of 1. From Theorem 1, page 138, it follows that to obtain P_x we have merely to multiply A_x by D_x/N_x. Hence

$$P_x = \frac{M_x}{N_x}. \tag{8}$$

Suppose it is desired to make n payments for the policy (if the insured lives to make them). The insurance is then called an *n-payment life* policy. The symbol ${}_nP_x$ is used for the *net annual premium.* From Theorem 2, page 138, it follows that to find the quantity ${}_nP_x$ we have merely to multiply A_x by $D_x/(N_x - N_{x+n})$. Hence

$${}_nP_x = \frac{M_x}{N_x - N_{x+n}}. \tag{9}$$

Example 2. Find the net annual premium on a whole life policy for $5000 on the life of a person aged 25.

In symbols the premium is

$$\$5000P_{25} = \$5000\frac{M_{25}}{N_{25}}.$$

From the table, $M_{25} = 15{,}145.7$, $N_{25} = 939{,}926$, so that the premium is

$$\frac{\$75{,}728{,}500}{939{,}926} = \$80.57.$$

Example 3. Find the net annual premium on a 20-payment life policy for $10,000 on the life of a person aged 35.

In symbols the premium is

$$\$10{,}000{}_{20}P_{35} = \$10{,}000\frac{M_{35}}{N_{35} - N_{55}}.$$

From the table, $M_{35} = 12{,}209.4$, $N_{35} = 579{,}161$, $N_{55} = 170{,}140$. Hence the premium is

$$\frac{\$122{,}094{,}000}{409{,}021} = \$298.50.$$

The first premium, in the case of annual premiums, is always paid at the time the policy is delivered.

EXERCISES

1. By (6), (4), and (2) show that

$$\begin{aligned} M_x &= v^{x+1}(l_x - l_{x+1}) + v^{x+2}(l_{x+1} - l_{x+2}) + \cdots \\ &= v(v^x l_x + v^{x+1} l_{x+1} + \cdots) - (v^{x+1} l_{x+1} + v^{x+2} l_{x+2} + \cdots), \end{aligned}$$

and therefore that

$$M_x = vN_x - N_{x+1}.$$

2. Find M_{40} by using the relation in Exercise 1.

3. By using (7) in the relation for M_x in Exercise 1 and the relation (6), page 134, obtain equation (1).

4. Find the net single premium for a whole life policy for $2500 on the life of a person aged 23.

5. Find the net annual premium for a life policy of $10,000 on a person aged 32.

6. Find the net annual premium for a 30-payment life policy for $5000 on the life of a person aged 27.

7. Let ${}_{r|}A_x$ denote the net single premium for an insurance of 1, deferred r years (that is, the benefit is paid only if the insured dies after age $x + r$). Show that

$${}_{r|}A_x = \frac{M_{x+r}}{D_x}.$$

8. Consider an insurance that pays 1 if the insured dies in the 1st year, 2 if he dies in the 2nd year, 3 if he dies in the 3rd year, etc. Let $(IA)_x$ be the net single premium for such an *increasing insurance.*[*] Show that

$$(IA)_x = A_x + {}_{1|}A_x + {}_{2|}A_x + \cdots + {}_{95-x|}A_x,$$

or

$$(IA)_x = \frac{D_x}{R_x},$$

where

$$R_x = M_x + M_{x+1} + M_{x+2} + \cdots \text{ to table limit.}$$

The value of R_x can be found in more extensive tables than those in this book.[†]

9. Some policies provide that upon death, in addition to 1, all premiums that have been paid by the insured are to be returned without interest. If P is the yearly premium, death benefits at the end of 1, 2, 3,$\cdots$, years are respectively

$$1 + P, \quad 1 + 2P, \quad 1 + 3P, \quad \cdots.$$

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[*] Increasing insurance has not been particularly popular. The needs of a person's beneficiaries do not usually increase steadily with his age. If a person wishes more protection for his dependents than they have had, he usually prefers to take out new insurance.

[†] *See* Glover, *Tables of Applied Mathematics in Finance, Insurance, Statistics,* George Wahr, Ann Arbor, 1923.

Show that the net single premium is

$$A_x + P(IA)_x.$$

The unknown yearly premium P enters here. From Theorem 1, page 138, it follows that the net single premium is also

$$P(1 + a_x).$$

Hence show that

$$P = \frac{A_x}{1 + a_x - (IA)_x} = \frac{M_x}{N_x - R_x}.$$

If the premiums are returned, what is it that supports the insurance of 1? Show that P increases indefinitely as i approaches 0 as a limit.

10. Find the annual premium of an insurance for $5000 with return of premium for a person aged 25, using $R_{25} = 458{,}928$. Compare with the premium in Example 2, page 149?

85. Term insurance. There are occasions when a person may desire insurance protection for a limited number of years. For instance he may be required to take out such a policy on account of some loan he has obtained for business reasons, the benefit being payable to the person's estate.

The symbol $A^{1}_{x\overline{n}|}$ represents the *net single premium* for a *term insurance* of 1 for n years on the life of a person of age x. The numeral 1 above the letter x implies that the life must expire before the period for the benefit to be paid.

It is clear that

$$A^{1}_{x\overline{n}|} = \frac{vd_x + v^2 d_{x+1} + \cdots + v^n d_{x+n-1}}{l_x}.$$

Proceeding as before, we find

$$A^{1}_{x\overline{n}|} = \frac{C_x + C_{x+1} + \cdots + C_{x+n-1}}{D_x}.$$

The numerator is merely

$$M_x - M_{x+n},$$

so that

$$A^{1}_{x\overline{n}|} = \frac{M_x - M_{x+n}}{D_x}. \tag{10}$$

The symbol $P^{1}_{x\overline{n}}$ is used for the net annual premium for an insurance of 1 for n years on the life of a person of age x. To find the annual premium we have by Theorem 2, page 138, to multiply the net single premium by $D_x/(N_x - N_{x+n})$.

Therefore

$$P^{1}_{x\overline{n}|} = \frac{M_x - M_{x+n}}{N_x - N_{x+n}}. \tag{11}$$

Let $n = 1$; then

$$A^{1}_{x\overline{1}|} = \frac{C_x}{D_x} = \frac{v^{x+1}d_x}{v^x l_x} = \frac{vd_x}{l_x}. \tag{12}$$

$A^{1}_{x\overline{1}}$ is sometimes called the *natural premium* and is the cost of an insurance of 1 on a person aged x for one year.*

EXERCISES

1. Find the annual premium for an insurance of \$3000 for 10 years on the life of a person aged 30.

2. Find the natural premium for a \$1000 policy on persons of age 25, 40, and 55.

3. Show that the cash sum a company must have to insure the l_x persons of age x for 1 for n years is

$$\frac{M_x - M_{x+n}}{v^x}.$$

4. Suppose that an n-year term insurance policy was written that provided that all death benefits be paid at age $x + n$ by the persons who survive to age $x + n$. Show that for an insurance of 1 each survivor should pay the premium.

$$_nk_x = \frac{M_x - M_{x+n}}{v^x}(1+i)^n \frac{1}{l_{x+n}} = \frac{M_x - M_{x+n}}{D_{x+n}}.$$

The quantity $_nk_x$ is called the *accumulated cost of insurance* and will be used presently in an exercise to evaluate the reserve on a policy. Could a policy such as that described have any practical value?

5. Consider a term insurance for n years to be paid for in r annual premiums, where r is less than n. Let the annual premium be $_rP^{1}_{x\overline{n}|}$ and prove that

$$_rP^{1}_{x\overline{n}|} = \frac{M_x - M_{x+n}}{N_x - N_{x+r}}$$

6. Accepting \$76.87 as the difference between the premiums of two \$5000 insurance policies at age 25, one providing for return of premiums and the other not, consider whether it is more advantageous to the beneficiary for the insured to take the first policy or the second, if in the latter instance the insured himself makes yearly deposits of \$76.87 in a fund that pays 3% interest (the beneficiary of the insurance being also the beneficiary of the deposits). Suppose the insured dies at age $x + n$; show that one is led to consider the equation

$$76.87 s_{\overline{n}|}(1.03) = 157.44n;$$

that is,

$$79.18 s_{\overline{n}|} = 157.44n, \text{ or } s_{\overline{n}|} = 1.9894n \ (3\%).$$

---

* The natural premium is also denoted by c_x and is given in the more extensive tables.

Show that $n = 43$ approximately satisfies the equation. Discuss the result.

7. Should life insurance companies require a medical examination before selling a policy? Compare with the question in Exercise 7, page 135.

86. Endowment insurance. If one owns a term insurance policy and also a pure endowment payable to the insured at the expiration of the insurance term, he has a combination of contracts that possesses attractive features. In the event of his death during the period in question his beneficiary receives the insurance; in the event of his survival, he himself is paid the endowment. When the two contracts are joined in one policy we have an *endowment insurance.*

The symbol $A_{x\overline{n}|}$ is used for the *net single premium* for an *n-year endowmeni* insurance of 1 on a person aged x. (The symbol is the same as that for term insurance except that the numeral 1, showing that the life must expire before the term if the benefit is to be paid, is now missing.) We have

$$A_{x\overline{n}|} = A^{1}_{x\overline{n}|} + {}_nE_x. \tag{13}$$

Using (10), and (4), Chapter IX, we have

$$A_{x\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x}. \tag{14}$$

The *net annual premium,* denoted by $P_{x\overline{n}|}$, the last payment to be made one year before the expiring of the insurance and the paying of the endowment, is obtained by multiplying the net premium by $D_x/(N_x - N_{x+n})$, and is

$$P_{x\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+n}}. \tag{15}$$

Example. Find the net annual premium for a 30-year endowment insurance for \$5000 on a person aged 25.

In symbols the premium is

$$\$5000 P_{25\,\overline{30}|} = \$5000\,\frac{M_{25} - M_{55} + D_{55}}{N_{25} - N_{55}}.$$

Using the table we have the following values for the numerator and denominator:

$$\begin{array}{rr@{\qquad}rr}
M_{25} = & 15{,}145.7 & N_{25} = & 939{,}926. \\
M_{55} = & 7{,}748.3 & N_{55} = & 170{,}140. \\
\cline{2-2}\cline{4-4}
 & 7{,}397.4 & \text{denom.} = & 769{,}786. \\
D_{55} = & 12{,}703.9 & & \\
\cline{2-2}
\text{num.} = & 20{,}101.3 & &
\end{array}$$

Hence the premium is

$$\frac{\$100{,}505{,}500}{769{,}786} = \$130.56.$$

It should be clearly realized that the payment that the insured receives in case he lives *is not the insurance.* The merit of an endowment insurance is that a savings contract is combined with an insurance obligation, so that the holder of the policy is under a very effective form of compulsion.

EXERCISES

1. Find the annual premium for a 20-year endowment insurance for \$5000 at age 28.

2. Find the annual premium for a 30-year endowment insurance for \$2000 at age 35.

3. Find the annual premium for a 5-year endowment insurance for \$1000 at age 25.

4. A man aged 30 has taken out a \$5000 20-year endowment policy; a second man also aged 30 has taken out a 20-year term insurance for \$5000, and plans to deposit the difference between the premiums for an endowment and the term insurance in a bank that pays 3% interest. Compare the amounts that the two men will receive if they live 20 years, and also the amounts they will leave if they die just before the expiration of 20 years.

5. If a person with an endowment insurance survives and receives the endowment, he is in a way the beneficiary of persons with similar policies who did not survive. On the other hand, if he does not survive, the beneficiary of his insurance receives no more than he would if the insured had had only a term insurance (and thus has paid a much lower premium). Show that if an endowment insurance were devised on a mere "savings basis," that is, if the excess of premiums above those for a term insurance were payable with interest to the beneficiary in addition to the insurance in the event of the insured's death, the annual premium would be

$$P^1_{x\overline{n}|} + \frac{1}{s_{\overline{n+1}|} - 1} = P^1_{x\overline{n}|} + \frac{1}{s_{\overline{n}|}}\,v.$$

6. Compute the annual premium on a 20-year policy for \$5000 taken out at age 30 on the basis of the last exercise, using 3% in $s_{\overline{n}|}$. What will the beneficiary receive if the insured dies at age 40?

7. Why is a whole life policy an endowment at age 96?

87. Reserves on policies. The cost of insuring a person for one year, that is, the natural premium, increases continually as one's age increases. On the other hand, the yearly premiums that have been obtained remain the same throughout the period of payment; that is, they are *level premiums*. An example will make the matter clearer.

The net premiums for an insurance of \$1000 for one year at ages 25, 40, and 55 are respectively (Exercise 2, page 152)

$$\$7.83, \quad \$9.51, \quad \$18.03.$$

On the other hand, the yearly premium on a whole life policy for \$1000 taken out by a person aged 25 is \$16.11, or one-fifth the premium in Example 2, page 149. Thus a person taking out the latter policy pays the company \$16.11 − \$7.83 = \$8.28 more the first year than is needed to meet the death claims against the company, while his payment when he has reached the age of 55 is too small to meet the death claims by \$18.03 − \$16.11 = \$1.92; as the age increases, the deficiency in the premium to meet the requirements of death claims increases rapidly.

It follows from what has been said that an insurance company must save out from the premiums it collects during the early years of a policy sums which will serve to carry the policy and assure the protection it guarantees during the later years when the premiums have become less than the natural premium for the insured at the age he has attained. Such sums are called in general *reserves*.* The reserves constitute the largest item of the various liabilities of an insurance company, and the major part of the assets are needed to cover them.

It is not difficult to start with a policy at the time it goes into effect and year by year trace the formation of the reserve. For the illustrative policy used earlier in this section there is at the end of the first year the amount

$$\$8.28(1.03)$$

to the credit of each person who paid the first premium, after all death claims have been paid. But the amounts for those who died during the year will be prorated to those who have survived. Thus the amount for each living policyholder is

$$\$8.28(1.03)\,\frac{l_{25}}{l_{26}} = \$8.28(1.03)\,\frac{89{,}032}{88.314} = \$8.60,$$

---

* It should be clearly understood that the word *reserve* is used in a different sense in life insurance than in banking. In insurance it denotes an actual *ascertained liability*, not a provision for a possible but unforeseen loss or emergency. In addition to its reserve, a life insurance company may have a surplus to meet emergencies or other types of expenses.

and this is the reserve on his policy. Accepting the fact that the natural premium at age 26 is \$7.89 we see that the reserve at the end of the 2nd year is

$$\left[\$8.28(1.03)\frac{l_{25}}{l_{26}} + (\$16.11 - \$7.89)\right](1.03)\frac{l_{26}}{l_{27}}$$

$$= \$8.28(1.03)^2\frac{l_{25}}{l_{27}} + \$8.22(1.03)\frac{l_{26}}{l_{27}}$$

and so on. It is not difficult to generalize the result, and this will be done later (Exercise 6, page 160).

What we desire is a formula by which the reserve at the end of any year can be computed. There are different ways of arriving at such a formula, and all of them have their merits. It is well to look at different procedures, for they help to develop accurate thinking on insurance theory. Two approaches to the problem will be given in the text; others occur in the exercises.

88. Formulas for reserves. We begin with the simplest case.

(1) Whole life policy with annual life payments. Let ${}_tV_x$ be the reserve on such an insurance for 1 taken out at age x, at the end of t years. It is the reserve just before the $(t + 1)$th premium is paid; that is, it is the *terminal reserve* for t years. Two derivations of ${}_tV_x$ follow.

(a) At the end of t years the yearly premium P_x is due, and the company can give to the premiums it is still to receive the value

$$P_x(1 + a_{x+t});$$

for the insured being of age $x + t$, an annuity due on his life with annual payment of 1 has the value $(1 + a_{x+t})$, so that an annuity due of P_x has the value set down above. Now it is clear that ${}_tV_x$ must be precisely the amount that, added to the present value of payments yet to be received, will give the net single premium required to insure a person of age $x + t$. That is,

$${}_tV_x + P_x(1 + a_{x+t}) = A_{x+t},$$

so that

$$(16) \qquad {}_tV_x = A_{x+t} - P_x(1 + a_{x+t}).$$

Using (7) and (8), and (8), page 134, we have

$$_tV_x = \frac{M_{x+t}}{D_{x+t}} - \frac{M_x}{N_x}\left(1 + \frac{N_{x+t+1}}{D_{x+t}}\right).$$

But by (6), page 134.

$$D_{x+t} + N_{x+t+1} = N_{x+t},$$

so that

(17) $$_tV_x = \frac{M_{x+t}N_x - M_xN_{x+t}}{N_xD_{x+t}}.$$

Example. What is the terminal reserve at the end of 20 years on a whole life policy for \$5000 taken out at age 30?

In symbols we have for the reserve

$$\$5000{}_{20}V_{30} = \$5000\frac{M_{50}N_{30} - M_{30}N_{50}}{N_{30}D_{50}}.$$

Using the values

$N_{30} = 742{,}484$ $M_{30} = 13{,}574.8$

$N_{50} = 243{,}156$ $M_{50} = 8{,}840.57$ $D_{50} = 15{,}922.8$

we have for the reserve

$$\$5000\frac{3{,}263{,}187{,}707}{11{,}822{,}424{,}235} = \frac{\$16{,}315.94}{11.822424} = \$1380.08.$$

The method of deriving $_tV_x$ given above is called the *prospective method*, since it is based upon premiums still to be paid, that is, in prospect.

(b) There is a different form of the prospective method that does not make use of the present value of a life annuity. The yearly premium that a person must pay for an insurance of 1 taken out at age $x + t$ (that is, at the age at which we are considering the reserve) is P_{x+t}. The premium the person is paying, namely P_x, will therefore support an insurance of P_x/P_{x+t} after he has attained the age $x + t$, instead of an insurance of 1. But the policy the person holds gives him an insurance of 1. Consequently the company must have accumulated from previous payments (by himself and others) a cash fund that will support an insurance of

$$1 - \frac{P_x}{P_{x+t}}.$$

But the cash amount necessary to support an insurance of 1 on a person at age $x + t$ being A_{x+t}, we have merely to multiply the quantity above by A_{x+t} in order to have the amount necessary to provide the insurance not covered by future premiums. Hence

$$(18) \qquad {}_tV_x = \left(1 - \frac{P_x}{P_{x+t}}\right) A_{x+t}.$$

When use is made of (7) and (8), the right-hand member of (18) reduces to that in (17).

In the exercises that follow, we consider the reserve by the *retrospective method.*

(2) Limited payment whole life policy. Let ${}_{t:n}V_x$ be the terminal reserve on an n-payment whole life policy for 1 taken out at age x, at the end of t years. After the premium due is paid, there remain

$$(n - 1) - t = n - t - 1$$

contingent premiums, so that the company can assign the value

$${}_nP_x(1 + a_{x+t:\overline{n-t-1}|})$$

to the premiums it is yet to receive before the premium due is paid. By the prospective method we then have

$$(19) \qquad {}_{t:n}V_x = A_{x+t} - {}_nP_x(1 + a_{x+t:\overline{n-t-1}|}).$$

By reasoning similar to that which gave (18) we have

$$(20) \qquad {}_{t:n}V_x = \left(1 - \frac{{}_nP_x}{{}_{n-t}P_{x+t}}\right) A_{x+t}.$$

An expression for ${}_{t:n}V_x$ in terms of commutation symbols is readily obtained. Leaving the annual payment in the formula, we have

$$(21) \qquad {}_{t:n}V_x = \frac{M_{x+t} - {}_nP_x(N_{x+t} - N_{x+n})}{D_{x+t}}.$$

(3) Endowment insurance. Let ${}_tV_{x\overline{n}|}$ be the terminal reserve at the end of t years for an n-year endowment insurance of 1 taken out at age x.

The formula analogous to (16) is readily seen to be

$$(22) \qquad {}_tV_{x\overline{n}|} = A_{x+t:\overline{n-t}|} - P_{x\overline{n}|}(1 + a_{x+t:\overline{n-t-1}|}),$$

while the formula analogous to (18) is

$$(23) \qquad {}_tV_{x\overline{n}} = \left(1 - \frac{P_{x\overline{n}}}{P_{x+t\,\overline{n-t}|}}\right) A_{x+t\,\overline{n-t}|}.$$

This can be written

$$(24)\qquad {}_tV_{x\overline{n}|} = \frac{M_{x+t} - M_{x+n} + D_{x+n} - (N_{x+t} - N_{x+n})P_{x\overline{n}|}}{D_{x+t}}.$$

EXERCISES

1. Use logarithms to obtain the result in the Example on page 137, by completing the following computation:

$$\begin{aligned}
\log M_{50} &= \\
\log N_{30} &= \\
\log M_{50}\,N_{30} &= 9.81\ 7167 \\
M_{50}N_{30} &= \\
M_{30}N_{50} &= \\
M_{50}N_{30} - M_{30}N_{50} &= 3.26\ 317 \times 10^{9}
\end{aligned}$$

$$\begin{aligned}
\log M_{30} &= \\
\log N_{50} &= \\
\log M_{30}\,N_{50} &= 9.51\ 8619 \\
\log 5000 &= \\
\log (M_{50}N_{30} - M_{30}N_{50}) &= \\
\operatorname{colog} N_{30} &= \\
\operatorname{colog} D_{50} &= 5.79\ 7971 - 10 \\
\log \text{Reserve} &= 3.13\ 9892 \\
\text{Reserve} &= \$1380.04
\end{aligned}$$

The use of subtraction logarithms would shorten the work, for they allow $\log (M_{50}N_{30} - M_{30}N_{50})$ to be found from $\log M_{50}N_{30}$ and $\log M_{30}N_{50}$ by one interpolation, instead of three, as used above.[*]

2. Find the 10th terminal reserve on a whole life policy for $2000 taken out at age 23.

3. Find the 5th terminal reserve for a $5000 20-payment life policy taken out at age 25.

4. Find the 15th terminal reserve for a $1000 30-payment endowment policy taken out at age 30.

5. If a person makes t payments of the annual premium P_x for a whole life policy, and if no death claims were paid from his premium or those of other survivors, show by Exercise 8, page 141, that he would be entitled at age $x + t$ to receive the amount

$$P_x \frac{N_x - N_{x+t}}{D_{x+t}}.$$

By Exercise 4, page 152, if the survivors at age $x + t$ pay the insurance benefits for all who have died, each should be charged the amount ${}_tk_x$. Hence show that

$${}_tV_x = P_x \frac{N_x - N_{x+t}}{D_{x+t}} - \frac{M_x - M_{x+t}}{D_{x+t}}.$$

---

[*] Addition and subtraction logarithms are given in Bremiker's *Tables Common Logarithms of Numbers and Trigonometrical Functions to Six Pl* *Decimals* (G. E. Stechert and Co., New York, 1923). This table should be hands of anyone having extensive logarithmic computations to mal logarithm of a number with five figures can be taken out without interpo

This is the *retrospective method* of calculating the reserve. Show that the result reduces to (17).

6. By generalizing the procedure used in section 87, show that

$$
\begin{aligned}
{}_nV_x = P_x[(1+i)^n l_x + (1+i)^{n-1} l_{x+1} + \cdots + (1+i) l_{x+n-1}] \frac{1}{l_{x+n}} \\
- [(1+i)^n A^{\;1}_{x:\overline{1|}} l_x + (1+i)^{n-1} A^{\;\;\;\;1}_{x+1:\overline{1|}} l_{x+1} + \cdots \\
+ (1+i) A^{\;\;\;\;\;\;\;1}_{x+n-1:\overline{1|}} l_{x+n-1}] \frac{1}{l_{x+n}}.
\end{aligned}
$$

Using the relations (obtained from (12)),

$$A^{\;1}_{x:\overline{1|}} = \frac{vd_x}{l_x}, \qquad A^{\;\;\;\;1}_{x+1:\overline{1|}} = \frac{vd_{x+1}}{l_{x+1}}, \qquad \text{etc.},$$

and other previous formulas, obtain the value for ${}_rV_x$ in Exercise 5.

7. Show that

$$l_{x+t+1} \cdot {}_{t+1}V_x = l_{x+t}({}_tV_x + P_x)(1+i) - d_{x+t}$$

is the equation by which the reserve for one year can be computed from the reserve of the previous year, each side of the equation representing the total amount of money in the reserve fund for the policies in question when those insured are of age $x + t + 1$.

Putting

$$u_x = \frac{D_x}{D_{x+1}}, \qquad k_x = \frac{C_x}{D_{x+1}},$$

show that

$${}_{t+1}V_x = u_{x+t}({}_tV_x + P_x) - k_{x+t}.$$

This is Fackler's *accumulation formula*. The quantities u_x and k_x are given in the more extensive tables.* Explain how a table of these quantities greatly expedites the computation of the reserve. Should the formula be used over a long succession of, say 20 years, or should (17) be used from time to time? Why?

8. The premium on a 5-year endowment insurance for \$1000 taken out at age 25 is \$186.24 (Exercise 3, page 154). Complete the following table and show that at the end of 5 years the fund is just sufficient to pay each survivor \$1000.00.†

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* Glover, *op. cit.*

† The premium carried to more places is \$186.2394; that is, the premium for a \$10,000 policy is \$1862.39. Using more places would affect the columns where large sums are involved, but not the column for the reserve to cents.

Direct multiplication is necessary, for logarithms will not give the number of figures shown unless a table with many places is used. The problem shows the great boon that calculating machines are for actuarial work. Five years are a rather short period for an endowment policy, but in order to show conclusively the series

Year	Persons Living	Premiums Received	Fund at Beginning of Year	Fund with 3% Interest	Death Claims	Fund at End of Year	Number of Survivors	Reserve per Policy
1	89,032	$16,581,319.68	$16,581,319.68	$17,078,759.27	718	$16,360,759.27	88,314	$185.26
2	88,314	16,447,599.36	32,808,358.63	33,792,609.39	718	33,074,609.39	87,596	377.58
3	87,596	16,313,879.04	49,388,488.43	50,870,143.08	718	50,152,143.08	86,878	577.27
4	86,878				718		86,160	
5	87,160				719		85,441	

9. Verify the 3rd reserve given in the table in Exercise 8 above by direct calculation by (22).

Partial results to be used are (as the student should verify):

$$A_{28.\overline{2}|} = .942\ 830, \qquad P_{25.\overline{5}|} = .186\ 240, \qquad a_{28\,\overline{1}|} = .962\ 857.$$

10. What should be the reserve on an n-payment life policy when $t = n$?

11. After the lapse of sufficient time, the level yearly premium being paid is less than the natural premium, so that the premiums paid are not sufficient to cover death losses, and the reserve must be drawn upon. Does this mean that the reserve decreases? Are there cases where the reserve, after increasing, ultimately decreases? Consider the case of a term insurance.

89. Surrender of policy. The reserve has been explained as a necessary means of guaranteeing the validity of the insurance when level premiums are used. We have seen that the payments above the natural premiums made in the earlier years are not used to augment the death payment; they must be put to the credit of the reserves of the survivors in order to meet requirements of future death claims. Thus the reserve on a policy is not like a bank account when an insured person dies. But obviously so long as a person survives he has an equity in the reserve on his policy. If by nonpayment of premiums he drops his policy, he is entitled to receive in some form a large share of his reserve.

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of reserves it is necessary to have either a term insurance or an endowment insurance of short period, if a long table is to be avoided.

It is rather customary for a person who surrenders a policy to have three choices as to how the reserve on it should be used:

(1) Cash payment. The insured may receive a cash payment, the so-called *cash surrender value.* This amount may not be the full amount of the reserve, the insurance laws of the different states allowing companies to make a surrender charge. The cash surrender value of a policy is the basis for loans that a company will make a policyholder.

(2) *Paid-up insurance*, for a portion of the original insurance amount. This option employs the cash surrender value as the net single premium on a whole life policy for the age which the insured has attained.

(3) *Extended insurance*, for the full amount of the original insurance for a limited period. This option uses the cash surrender value as the net single premium for a term insurance.

Details as to the three plans, that is, statements as to the cash surrender value, the amount of paid-up insurance, and the length of extended insurance, for different policy years, are stated on insurance policies. It is usually provided that if an insured stops paying premiums without informing the company as to the option he has elected, the company will apply option (3).

EXERCISES

1. If the surrender value of a policy is $1728, what paid-up whole life policy should be given a man aged 45 if he surrendered his original policy?

2. A man aged 52 wishes to stop paying premiums on an insurance policy. What paid-up whole life policy should he receive if the surrender value of his policy is $5218?

3. A man aged 50 is carrying a policy for $5000. If he stops paying premiums what should be the period of extension if the surrender value of the policy is $1380?

SUGGESTION: From (10) we have

$$5000\frac{M_{50}-M_{50+n}}{D_{50}}=1380,$$

where n is the period of extension. From this find M_{50+n}, and then n.

4. A man aged 40 is carrying a $10,000 policy. What extended insurance should be offered him if the policy reserve is $1525?

90. Dividends. The most essential feature of an insurance policy should be that the benefits will be paid without fail. Thus premiums must be set sufficiently high to guarantee the soundness of the policy, and to this end it is assumed that the company earns a relatively low rate of interest on investments. For years the rate of 3½% was very generally assumed. Beginning about 1935 the rate was lowered to 3% or even to a smaller figure, largely on account of the great decrease in the volume of sound bonds paying interest at rates formerly customary. The change of course caused a rise in the premiums that companies charged for their new policies.

In addition to earning a somewhat higher rate on its loans and investments than it uses in computing cost of insurance, companies also profit by using the American Experience Table. This table shows a much greater mortality at lower ages than that now experienced, a fact that adds additional security to insurance policies.

In the case of a mutual company all the profits belong to the policyholders, for the policyholders are the company. After all death claims have been paid, operating expenses discharged, the reserve carefully calculated, and provision made for some surplus,* the balance is the profit to be distributed to policyholders in the form of dividends.† The statement of the dividend on his policy is usually sent to a policyholder with the notice of his premium. He may use the dividend to reduce his payment to the company, or allow it to remain with the company, where a stated interest rate is applied to it. Dividends so left increase the death benefit paid the beneficiary, or the cash surrender value payable to the policyholder.

A policy that yields dividends to the policyholder is called a *participating policy*.

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* The surplus may be used to meet such charges as the agent's commission and the medical examiner's fee. The expenses may, in fact, take a large proportion of the first premium.

† The gross premiums that different companies quote for policies essentially similar are often appreciably different. It is not possible to make a thorough comparison without going into the dividend record of the companies, a thing a little difficult for the prospective buyer to do. One company may be charging larger premiums than another, but returning more in the way of dividends. But it is to be noted that there is no guarantee as to dividends.

In contrast to a mutual company, a stock company operates to make profit for the stockholders, the stock of the company being a guarantee behind the policies that the company writes. Such companies generally issue *nonparticipating policies*. To compete with mutual companies selling participating policies, the stock company must normally keep the gross premiums at which it sells its policies somewhat below the premiums of the mutual companies, which return excess premiums through the dividends.

91. Remarks. In the last two chapters we have given an exposition of the foundations of actuarial theory, limiting the applications to the more common types of contracts. The student should not think of the subject as a completed one, nor believe that actuaries limit themselves to application of ideas already developed. There is a constant effort to increase the field of usefulness of insurance and annuities, and to provide new benefits that will be attractive to the public and be actuarially sound.

A very interesting and important question is that of insurance laws. States generally have such laws, and as companies wish to sell insurance over wide areas they cannot be governed solely by the laws of the state in which their home office is located. States do not generally fix directly the mortality table or the rate of interest to be used in computing premiums, so that effective competition exists between companies. States do, however, protect policyholders by requiring companies to have proper reserves for all policies, and by making provision as to how reserves shall be determined they indirectly but substantially influence premium calculations.*

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* The following books are recommended for further study:

Menge, Walter O., and Glover, James W., *An Introduction to the Mathematics of Life Insurance*, New York, The Macmillan Co., 1935. As the title indicates, this book deals especially with actuarial theory.

Maclean, Joseph B., *Life Insurance*, 6th edition, New York, McGraw-Hill Book Co., 1945. A critical discussion of life insurance, but not an actuarial treatment.

SUPPLEMENTARY EXERCISES

Simple Interest and Simple Discount

1. What is the exact interest charge on a principal of $1372 invested at a rate of 5½% from March 3 to the following November 23?

2. A loan of $2000 is made on January 1 at 7% ordinary interest. On what day will $2075 or more first be due?

3. The ordinary interest on a certain principal is $373. Find the exact interest on the same principal.

4. Find the length of time in which any sum will double itself at a rate of 3½% ordinary interest.

5. What principal must be invested at 4% simple interest to amount to $10,000 in 3 years?

6. Compare the ordinary interest with the exact interest on a note for $800 dated September 2, 1931, and due May 6, 1932, if the ordinary interest amounts to $31.64.

7. A man received two offers for a farm that he had for sale. A offered him $1500 down and $1500 in one year; B offered him $1000 down, $1000 in 6 months, and $1000 in one year. With the prevailing rate of interest at 7% which offer was the better, and by how much?

8. What is the discount on $3600 for 9 months at a discount rate of 7%?

9. A 7 months' 5½% interest-bearing note dated July 1 was discounted October 12 at 7%. The proceeds amounted to $1774.26. What was the face value of the note?

Consider the year as having 365 days, and months as calendar months.

10. What discount rate is equivalent to a 7% ordinary interest rate on a 45-day note?

11. A bank charges 7% in advance. For what amount should a note be drawn to have proceeds of $1000 on a loan for 120 days, regarding the year as having 360 days?

12. On what date must a note for $150 due September 1 be discounted at a rate of 6% in order to obtain $145 as the proceeds if 365 days are used for the year?

13. What interest rate is equivalent to a 6% discount rate in finding the present value of a note due in 9 months?

14. A bank uses a 7% discount rate. A noninterest-bearing note for $500, due in 60 days, was discounted on a certain day. Thirty days later the note was rediscounted at a Federal Reserve Bank, and a profit of $4.16 was realized by the first bank. What rediscount rate was used?

15. The proceeds from a $500 noninterest-bearing note were $496. The note was discounted at a rate of 6%. How many days before maturity was the note discounted if 365 days were used for the year?

16. Notes due in 6 months are discounted at 3%. Is double the corresponding interest rate the same as the interest rate that corresponds to a 6% discount rate? Discuss the general problem.

17. How much is saved at the end of one year by borrowing $3000 today at 6.2% simple interest instead of borrowing it from a man charging 6% simple discount?

Compound Interest and Present Value

18. What will a sum of $500 amount to in a period of 9 years if the interest is allowed to accumulate at the rate of 3½% per annum?

19. A bank deposit of $2500 was made on September 1, 1923. Withdrawals of $250 were made September 1, 1928, and September 1, 1933. If the bank paid 3%, what was the amount of the deposit on September 1, 1935?

20. What is the difference between the amount of $500 at simple interest and at compound interest for 10 years at 4%?

21. Find the amount of $1500 for 7½ months at 2%.

22. How long does it take a sum of money to triple itself at 3½%?

23. In 5 years a sum of $500 accumulated to $750. What rate of interest was used?

24. What is the amount of $560 for 3¾ years compounded every 3 months at a nominal rate of 2%?

25. Find the nominal rate if the effective rate is 5%, interest being converted continuously.

26. What nominal rate converted continuously is equivalent to 6% nominal converted monthly?

27. What nominal rate of interest convertible monthly is equivalent to a simple interest rate of 6% for a period of 4 years?

28. Find the effective rate corresponding to 4% convertible bimonthly.

29. What is the force of interest corresponding to 4% effective?

30. Should a merchant borrow money at a nominal rate of 8% in order to pay cash when he buys goods on 90-days credit or 2% off for cash?

31. What effective rate of interest corresponds to a discount of 3% for cash if 90 days are allowed in which to pay a bill?

32. What is the value of $1000 at the end of 8 months if the interest is convertible continuously at a nominal rate of 4½%?

33. What is the present value of a $4500 noninterest-bearing note due in 10 years if money is worth 5½%?

34. How much must be deposited in a bank paying 3½% effective interest compounded semiannually in order to have $5000 in 18 years?

35. A note of $200, due in 2 years, bears interest at 6% compounded bimonthly. What is its present value on a basis of 4½% convertible every 4 months?

36. How long will it take \$1000, accumulating at 3% nominal converted semiannually, to amount to \$2380.75?

37. A man buys a house by paying \$2000 down and \$2000 in 3 years. Interest is paid at 6% each year on the unpaid part. What is the sum of the present values of all payments at the same rate at compound interest?

38. Find the value of \$5000, due in 6 years with interest at 5% effective, at the end of 3 years if money is worth 4% convertible quarterly.

39. A man wishes to receive \$1500 at the end of each year for 5 years beginning 10 years from now. At what time could \$7500 be paid to be equivalent to the separate payments if money is worth 5½% effective?

40. The sum of \$3000 is due in 3 years and \$2000 in 4½ years. At the end of 2 years \$2500 is paid. What equal payments must be made at the end of 3½ years and of 4 years to cancel the remainder of the obligations if money is worth 6% compounded semiannually?

41. A man receives two offers for a farm. One offer is \$1000 down and \$3000 per year with interest at 5% for 3 years, the first payment due at the end of one year. The second offer is \$3000 down, and \$3000 in 2 years, \$3000 in 4 years, and \$2000 in 5 years with no interest. Which offer is the better and by how much if money is worth 6%?

42. The sum of \$6000 was paid for four notes as follows: \$1500 due in 2 years with 5% interest, \$1000 due in 3 years with 6% interest, \$2000 due in 3½ years with 7% interest compounded semiannually, and \$2000 due in 6 years with 5½% interest. If money is worth 4½%, what profit was realized at the time of purchase?

43. Find the value at the end of 3¼ years of \$7634.50 due in 8½ years with interest at 5% convertible quarterly if money is worth 6% convertible quarterly.

44. A father invested \$20,000 at 6% convertible semiannually at the time of the birth of his second son. If the first son was born 3 years previous to the second and a third son is born 5 years after the second, what sum could be given to the eldest son at the age of 21 so that the second and third sons could each receive a like sum at age 21?

45. If money is worth 5%, at what time could the total of the following notes be paid in a single payment: \$425 due in 30 days, \$375 due in 60 days, \$500 due in 90 days, and \$125 due in 120 days, none of the notes bearing interest.

46. How much will an owner lose by accepting an offer of \$1000 a year for 3 years for a house worth \$3000 now? The interest rate is 6% compounded semiannually.

Annuities

47. What is the difference in value at the end of 10 years between \$10,000 accumulated at 3% interest and \$1200 a year for 10 years with interest at 6%?

48. Find the amount of an annuity of \$500 for 10 years (a) with effective rate of 4%; (b) with nominal rate of 4% converted quarterly.

49. Prove that the amount is increased by $is_{\overline{n}|}$ if the annuity is paid at the beginning of the year instead of the end.

50. The sum of $200 is invested at the end of each year. At the end of 10 years $2346 has been accumulated. What rate of interest was used?

51. Find the amount of an annuity of $2000 per year payable quarterly for 20 years, interest at 6% converted quarterly.

52. Find the value of the series of payments in the last exercise at the time of the first payment.

53. Find the difference between the present values of two annuities each of $500 per month, one payable for 5 years, the other for 4 years 11 months, interest 6% converted monthly.

54. The present value of an annuity of $2500 per annum is $16,158. If money is worth 5% effective, how many payments are to be made?

55. A man desires to accumulate $6800 by depositing $500 annually in a bank paying 2%. When will the amount be reached? What final partial deposit would be needed to give $7200?

56. A man buys a lot for $200 down and $50 at the end of each month for 2 years. What would be the cash price if money is worth 6%, (a) effective? (b) nominal?

57. A man wishes to donate $15,000 to charity. He plans to accumulate the amount during a period of 5 years by equal semiannual deposits with a trust company paying 3% nominal compounded semiannually. When one-half of the payments have been made, the company changes the interest rate to 2% compounded semiannually. What new payment must be made after the change in interest rate in order that $15,000 may be accumulated in the time originally intended?

58. Is it better to pay $20,000 cash, or $10,000 cash and 8 yearly payments of $2000 each, the first payment at the end of 3 years? Money is worth 6% compounded semiannually.

59. A radio sells for $200 cash or on the following terms: $10 on the first of each month for one year; 6% of $200 is then added to the sales price and the first year's payments deducted. This amount is increased by 6%, and the result is paid in equal monthly installments during the second year. What rate of interest does the purchaser really pay?

60. A man aged 35 invests $25,000 in an annuity payable monthly for 20 years, beginning at the end of the first month after reaching the age of 55. What is the amount of the monthly installment if money is worth 6% effective?

61. Prove by using the basic formulas that

$$(1 + i)s_{\overline{n}|} = s_{\overline{n+1}|} - 1.$$

62. A man buys a house for $5000 on the first day of a month. He agrees to pay $50 on the 15th day of each month. If money is worth 5% effective, how long from the date of sale will it be until the house is paid for?

63. A man wishes to make equal bimonthly deposits in a bank for a period of 4 years, such that each year thereafter he may withdraw $500 for 4 years, the first withdrawal being made 6 months after the last deposit. The bank pays 3% nominal a year. What is the amount of each deposit?

64. If 12 annual payments of $1000 are made to pay a $11,000 debt, the first payment being made at once, what rate of interest is used?

65. A father leaves $25,000 to three sons aged 7, 10, and 11 years, respectively. At the age of 21 each son is to receive a 10-year annuity. If the money draws interest at 3½%, how much does each son receive per year?

66. Find the present value of an annuity of $1500 a year payable monthly for 10 years if money is worth 4% nominal converted semiannually.

67. A man agrees to repay a loan of $6000, with interest at 6% converted semiannually, by payments of $600 quarterly beginning at the end of 2 years. How long will it take to discharge the obligation, and what will be the amount of the last payment?

68. The sum of $8200 was paid for an annuity of $630 payable semiannually for 10 years beginning in 6 months. What rate of interest converted semiannually is received on the investment?

69. What monthly deposits in a bank paying 2% nominal compounded quarterly must a man make in order to buy a car worth $850 at the end of 1½ years?

70. A man has $1500 in a bank paying 3%. He will be admitted to a partnership in his firm as soon as he can increase his savings to $5000 How much must he deposit in the bank at the end of each 6 months in order to have $5000 at the end of 5 years? If he makes no deposit during the first year, what is the amount of the semiannual deposits during the remaining 4 years if $5000 is to be accumulated?

71. A man wishes to establish a fund to provide for the education of his son. His son is to receive $75 on the first of each month for 7 years beginning 8 years from now. If money is worth 4% compounded semiannually, how much must the father invest at the beginning of each year for 6 years to establish the fund?

72. Annual deposits of $100 each were made in a bank paying 3%, the last deposit being made January 1, 1925. Withdrawals of $500 each were made January 1, 1926, 1927, 1928, 1929, the balance after the last withdrawal being $1.51. How many deposits were made?

73. State a problem for which the equation of value would be

$$500s_{\overline{n+3}|} = 1500a_{\overline{3}|} + A,$$

where A is a known number. Can A be given arbitrarily?

74. State a problem for which the equation of value would be

$$400s_{\overline{n+8}|} = 700s_{\overline{8}|} + A - 300(1.04)^4, \quad (4\%),$$

where A is given. Change the sign of the last term and state a problem.

75. A coal mine yields $25,000 a year for 16 years and is then worth $75,000. What is its value if money is worth 4½%?

76. Annual deposits of \$500 each were made in a bank paying 4%, the last deposit being made January 1, 1920. Withdrawals of \$1000 each were made January 1, 1921, 1922, 1923, the balance after the last withdrawal being \$8140.30. How many deposits had been made?

If January 1, 1920, is taken as the time for writing the equation of value, show that

$$500s_{\overline{n}|} = 1000a_{\overline{3}|} + 8140.30v^3. \quad (4\%)$$

If January 1, 1923, is taken, show that

$$500(s_{\overline{n+3}|} - s_{\overline{3}|}) = 1000s_{\overline{3}|} + 8140.30, \quad (4\%)$$

or

$$500s_{\overline{n+3}|} = 1500s_{\overline{3}|} + 8140.30. \quad (4\%)$$

Complete the solutions and compare from the standpoint of computation.

77. State a problem for which the equation of value would be

$$400s_{\overline{n+5}|} = 1000s_{\overline{5}|} + A, \quad (3\tfrac{1}{2}\%)$$

where A is a known number. Could A be given arbitrarily?

78. The sum of \$500 was paid annually for 5 years into a fund earning 4%; then \$800 was paid annually. When did the fund amount to \$12,000?

SUGGESTION: If n payments of \$800 were made, show that

$$500s_{\overline{5}|} + 800a_{\overline{n}|} = 12{,}000v^n.$$

Using the expression for $a_{\overline{n}|}$ show that

$$500s_{\overline{5}|} + 800\,\frac{1 - v^n}{i} = 12{,}000v^n,$$

and thus obtain

$$32{,}000v^n = 500s_{\overline{5}|} + 20{,}000. \quad (4\%)$$

Hence obtain

$$v^n = .70963. \quad (4\%)$$

Show that $n = 8$; and thus obtain for the amount after 8 payments of \$800,

$$K = \$500(s_{\overline{13}|} - s_{\overline{8}|}) + \$800s_{\overline{8}|} = \$11{,}077.69.$$

Obtain the last partial payment.

79. In Exercise 16, when did the account amount to \$12,500 if simple interest was used for the fraction of a year?

80. The sum of \$300 was deposited in a fund earning 3½% for 4 years, then \$500 for 6 years, then \$700 until the fund amounted to \$10,000. How long was required?

81. It is desired to make yearly deposits of \$800 for a certain number of years, then yearly deposits of \$1200 for the same number of years, in a fund earning 4%, and to select the number of years so that \$12,000 will be accumulated. How many years will be required?

SUGGESTION: Let 2 n be the total number of deposits. Show that

$$800(s_{\overline{2n}|} - s_{\overline{n}|}) + 1200s_{\overline{n}|} = 12{,}000.$$

Obtain a quadratic equation in $(1.04)^n$, and thus find n.

Sinking Funds and Amortization

82. What is the monthly payment necessary for the amortization of a debt of $20,000 in 10 years, interest being 3% effective?

83. A man arranges to amortize a debt of $10,000 in 18 semiannual payments with interest at 5% converted semiannually. After 9 payments are made he arranges to pay the balance in 3 additional yearly payments. What is the amount of the new payments?

84. The sum of $5000 is due in 10 years with interest at 7% compounded semiannually. What annual payment must be made in a sinking fund to retire the debt when due if the deposits draw 4% compounded semiannually?

85. A college student borrows $500 each year for 4 years at the effective interest rate of 5½%. He wishes to repay the loans, principal and interest, in 10 annual payments beginning one year after graduation. What is the amount of the annual payment?

Note. Assume the student borrows each year on September 15 for the coming year, and that he graduates on June 15.

86. A debt of $10,000 is being amortized by a series of quarterly payments for a period of 10 years. (a) What is the quarterly payment, interest being 6% converted semiannually? (b) Find the outstanding liability just after 15 payments have been made.

87. Construct an amortization schedule for the last exercise for the last five years.

88. Find the total annual payment into a sinking fund accumulating at 4% for the retirement in 7 years of a loan of $5000 bearing interest at 5% if the interest is allowed to accumulate.

89. A debt of $50,000 is to be retired, principal and interest, in 20 equal semiannual installments. The unpaid principal bears 5% payable semiannually for 5 years, 6% semiannually for the second 5 years, and 7% semiannually for the last 10 years. Find the value of each payment.

90. Find the required semiannual payment into a sinking fund drawing 3% interest convertible semiannually to meet an obligation of $10,000 due in 5 years. Construct a schedule.

91. Construct a schedule for the retirement in 10 years of 100 bonds, par value $100, bearing 5% interest, payable semiannually. Bonds are callable at 105 in the open market. Arrange the schedule so that the sum of the amount of the semiannual interest and the amount paid for the bonds shall be as nearly constant as possible.

92. A man purchases a piece of real estate for $10,000, paying $3000 down and $50 at the end of each month until the debt is canceled. How long will it take to cancel the debt, interest being 6% effective?

93. The sum of $1000 is paid at the end of odd years and $2000 at the end of even years on an obligation of $20,000 with interest at 6% effective. What is the amount of the unpaid principal after the payment of $2000 at the end of the 6th year?

94. Construct a schedule for the last exercise, showing principal and interest paid annually.

95. A state levies a gasoline tax to provide funds for the retirement of a \$250,000,000 road bond issue bearing 6% payable semiannually in 15 years. Funds are to be placed in a sinking fund drawing 4½% compounded annually. How many gallons of gasoline must be sold annually if the tax amounts to 2.6 cents per gallon?

96. A man can borrow \$10,000 with interest at 6% payable semiannually, the principal being due at the end of 10 years and to be paid by the accumulation of a sinking fund drawing 5% converted semiannually. He also can borrow \$10,000 due in equal semiannual payments for 10 years. What rate of interest could he afford to pay in this latter case?

97. State problems for which the answers would be

$$\frac{\$75{,}000}{s_{\overline{30}|}}\,s_{\overline{18}|}(1.04)^4,\quad (3\%);\qquad \$50{,}000\left(1-\frac{s_{\overline{10}|}}{s_{\overline{15}|}}\right),\quad (4\%).$$

Bonds

98. A bond for \$1000 due in 20 years pays \$30 interest semiannually. If it is bought to yield 5% converted semiannually, what is the purchase price? What is the purchase price if the bond is redeemable for \$1050 and bought for the same yield?

99. Four \$5000 bonds bearing 5% interest payable January 1 and July 1 and maturing July 1, 1925, July 1, 1930, July 1, 1935, and July 1, 1940, respectively, were purchased on July 1, 1920, at prices yielding 6%. Compare the purchase prices.

100. Find the purchase price of a \$10,000 bond paying dividends semiannually to the amount of \$200, due in 20 years at 109, if the investment rate is 3% converted semiannually.

101. What is the difference in the purchase price of 2 bonds of \$500 each due in 15 years paying interest semiannually to the amounts of \$15 and \$10 respectively? The investment rate is 5% converted semiannually.

102. A \$10,000 bond due in 20 years on January 1, 1945, and paying 5% interest yearly on January 1 and July 1, was purchased April 1, 1927, to yield 6% converted semiannually. What was the purchase price?

103. A \$5000 5% bond will be redeemed with a premium at the end of 10 years. What is the redemption value if the purchase price is \$5500 to yield 4% annually?

104. Construct a schedule of a bond issue of \$20,000, of which \$6000 is redeemable at the end of 3 years and \$14,000 at the end of 6 years. The dividend rate is 5% and the investment rate 4%, both converted semiannually.

105. What is the yield of a \$1000 5% bond payable in 5 years bought for \$970?

106. Construct an amortization schedule for a \$1000 6% bond, interest payable semiannually, redeemable in 10 years for \$1090 and purchased at a price to yield 4% payable semiannually.

107. An issue of bonds matures in 20 years. No interest is paid for 5 years, but 8% annual dividends are paid thereafter. What is the purchase price per $1000 to yield a rate of 7%? Construct a schedule.

108. An issue of $2,000,000 worth of 3% bonds, paying dividends semiannually, is redeemable at par as follows: $100,000 at the end of 5 years, $200,000 at the end of 10 years, $400,000 at the end of 15 years, $800,000 at the end of 20 years, and $500,000 at the end of 25 years. The sum of $1,900,000 has been paid for the issue. What rate of interest will be realized on the investment?

109. A $5000 6% bond, interest semiannually, maturing in 10 years, sold for $5900. A $5000 5% bond, interest annually, maturing in 25 years, sold for $5500. What difference was there in the effective rates of interest?

110. Find the nominal yield rate converted semiannually of a $1000 4% bond paying interest semiannually, redeemable at 110 in 20 years, if it is purchased at (a) 80; (b) 120.

111. What premium will be paid for a $500 bond paying 4% interest semiannually, bought to yield 3½% converted semiannually and purchased 7½ years before the bond is due?

Depreciation and Replacement

112. A building costs $50,000 and lasts for 20 years, with a final salvage value of $5000. (a) Find the amount that should be in a sinking fund at the end of 10 years, interest being at 4½%. (b) What is the amount of the depreciation during the 11th year?

113. A certain industry owns three classes of equipment. The annual replacement charge is figured on a 4% basis under the sinking fund method. Class I of the equipment cost $100,000, salvage $10,000, life 20 years; class II cost $75,000, salvage $5000, life 25 years; class III cost $27,500, no salvage value, life 10 years. What is the annual charge for replacement?

114. An automobile costs $1000, lasts for 12 years, and has a final value of $200. Construct a schedule using the constant percentage plan.

115. A building requires $10,000 per annum for upkeep and repairs for a period of 10 years and thereafter $15,000 annually. What sum would be required to provide for perpetual upkeep, money being worth 5%?

116. The sum of $5000 is required annually to keep up a certain length of railroad. If money is worth 4½%, what sum could be expended in order to make improvements so that maintenance would be reduced to $3500 a year?

117. A bus costs $10,000 and will last for 5 years. Its salvage value will be $1200. What will be the book value the 4th year under the sinking fund plan with interest at 4%? How much will be in the fund at the end of 3 years?

118. Five hundred acres of timber will yield $2000 per acre annually for 15 years. The land will then be worth $100 per acre. What is the value of the tract of land if the yield rate is 8% and a sinking fund is accumulated at 3½%?

119. Should a business firm erect a building for $40,000 that will last 20 years or would it be more profitable to erect one costing $45,000 that will last 30 years, if money is worth 5%?

120. If it cost $2000 a year to maintain a number of athletic courts, what sum could be expended to improve the courts so that $500 would be saved in upkeep yearly, interest being 5%?

121. An electric plant costs $15,000,000. Operating expenses and upkeep amount to $1,500,000 per annum, and depreciation is figured at 5% yearly. The average output is 35,000 kilowatts. The plant is financed by equal amounts of stock and 4½% annual 20-year bonds. (a) What charge per kilowatt hour must be made to earn 7% on the first investment not including operation and upkeep? (b) Find the annual sinking fund charge for depreciation if accumulated at 5½%, and it is contemplated to rebuild the plant at the end of 20 years. (c) What dividend could be paid on the stock after paying out of the 7% income the depreciation charge and the bond interest?

APPENDIX

APPENDIX

I. THE BINOMIAL THEOREM

1. The binomial coefficients. If r is a positive integer, the symbol $r!$, read "factorial r," is defined by

$$r! = r(r-1)\cdots 3\cdot 2\cdot 1; \tag{1}$$

that is, $r!$ is the product of all the integers from 1 to r, inclusive.

Examples. $3! = 3\cdot 2\cdot 1 = 6; 5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120.$

For the sake of consistency in various formulas we also take

$$0! = 1. \tag{2}$$

The symbol ${}_nC_r$, where n and r are positive integers (r may be 0) and $n \geqq r$, is defined as

$${}_nC_r = \frac{n!}{r!(n-r)!}. \tag{3}$$

Example.

$$\begin{aligned}{}_{13}C_4 &= \frac{13!}{4!9!} = \frac{13\cdot 12\cdot 11\cdot 10\cdot 9\cdots 2\cdot 1}{(1\cdot 2\cdot 3\cdot 4)(1\cdot 2\cdot 3\cdots 9)}\\ &= \frac{13\cdot 12\cdot 11\cdot 10}{1\cdot 2\cdot 3\cdot 4} = 715.\end{aligned}$$

It is seen in the example just given that the quantity $(n-r)!$ in the denominator of ${}_nC_r$ can always be canceled into the numerator, leaving r factors. This fact simplifies the evaluation.

Example. ${}_{20}C_5 = \dfrac{20\cdot 19\cdot 18\cdot 17\cdot 16}{1\cdot 2\cdot 3\cdot 4\cdot 5} = 15{,}504.$

We have the important relation

$${}_nC_{n-r} = {}_nC_r. \tag{4}$$

For by (3) we have

$${}_nC_{n-r} = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!} = {}_nC_r.$$

The relation (4) can profitably be used to evaluate ${}_nC_r$ whenever $r > n/2$.

Example. ${}_{10}C_7 = {}_{10}C_3 = \dfrac{10\cdot 9\cdot 8}{1\cdot 2\cdot 3} = 120.$

2. The binomial theorem. It is shown in books on algebra that when n is a positive integer,

$$(5)\quad (a+x)^n = a^n + {}_nC_1a^{n-1}x + {}_nC_2a^{n-2}x^2 + \cdots + {}_nC_{n-1}ax^{n-1} + x^n.$$

Example. Expand $(a+x)^5$

Using (5) we have

$$\begin{aligned}(a+x)^5 &= a^5 + {}_5C_1a^4x + {}_5C_2a^3x^2 + {}_5C_3a^2x^3 + {}_5C_4ax^4 + x^5\\ &= a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5.\end{aligned}$$

The binomial theorem is especially useful in compound interest in obtaining such an expression as $(1.02)^8$ correct to a few decimal places.

Example. Obtain $(1.02)^8$ correct to 3 decimals.

We have

$$\begin{aligned}(1.02)^8 = (1+.02)^8 &= 1 + {}_8C_1(.02) + {}_8C_2(.02)^2 + {}_8C_3(.02)^3 + \cdots\\ &= 1 + 8(.02) + 28(.0004) + 56(.000\,008) + \cdots\\ &= 1.172.\end{aligned}$$

3. Nonintegral index. Let us take the case $a = 1$, and in (5) insert the values

$${}_nC_1 = n,\quad {}_nC_2 = \frac{n(n-1)}{1\cdot2},\quad {}_nC_3 = \frac{n(n-1)(n-2)}{1\cdot2\cdot3},\quad \cdots.$$

We then have

$$(6)\quad (1+x)^n = 1 + nx + \frac{n(n-1)}{1\cdot2}x^2 + \frac{n(n-1)(n-2)}{1\cdot2\cdot3}x^3 + \cdots,$$

the unwritten coefficients being easily formed.

The expansion (6) is valid when n is not a positive integer, provided that

$$(7)\quad -1 < x < 1;$$

that is, if x is numerically smaller than 1. The expansion does not terminate, but with the restriction on x that is imposed by (7) the terms approach zero and the series is what is called a *convergent* series.

Example. Expand $(1+x)^{1/2}$ to 5 terms.

We have, substituting in (6),

$$\begin{aligned}(1+x)^{1/2} &= 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1\cdot2}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1\cdot2\cdot3}x^3\\ &\quad + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{1\cdot2\cdot3\cdot4}x^4 + \cdots\\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \cdots.\end{aligned}$$

The expansion (6) for fractional values of n can be used to extract roots.

Example. Find $\sqrt{11}$ to 3 decimals.

We have $\sqrt{11} = \sqrt{9+2} = 3\sqrt{1+\frac{2}{9}} = 3\left(1+\frac{2}{9}\right)^{1/2}$.

Using the preceding expansion we have

$$\begin{aligned}\left(1+\tfrac{2}{9}\right)^{1/2} &= 1 + \tfrac{1}{2}\cdot\tfrac{2}{9} - \tfrac{1}{8}\cdot\tfrac{4}{81} + \tfrac{1}{16}\cdot\tfrac{8}{729} - \tfrac{5}{128}\cdot\tfrac{16}{6561} + \cdots \\ &= 1 + .11111 - .00617 + .00069 - .00010 + \cdots \\ &= 1.10553.\end{aligned}$$

Hence, $\sqrt{11} = 3.3166$.
The result is correct to the number of places shown.

An important case is that where $n = -1$. We find (as the student should show)

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots . \tag{8}$$

Replacing x by $-x$ in (8) we have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots . \tag{9}$$

Both (8) an (9) can be obtained by division without the Binomial Theorem. See Exercise 31, page 180.

Example. Evaluate $\frac{1}{.98}$.

$$\begin{aligned}\text{We have } \frac{1}{.98} = \frac{1}{1-.02} &= 1 + (.02) + (.02)^2 + (.02)^3 + \cdots \\ &= 1.020408163264.\end{aligned}$$

EXERCISES

Evaluate:

1. ${}_6C_2$. **2.** ${}_8C_5$. **3.** ${}_{17}C_{14}$.

Expand:

4. $(2x^2 - 3y)^4$. **5.** $(x - 2y)^6$. **6.** $\left(2x - \frac{1}{2y}\right)^5$. **7.** $(a + x)^7$.

Use the binomial theorem to obtain:

8. $(102)^3$. **9.** $(99)^4$. **10.** $(1001)^5$. **11.** $(998)^4$.

Obtain to 4 decimals:

12. $(1.01)^8$. **13.** $(1.03)^7$. **14.** $(1.005)^6$. **15.** $(1.025)^5$.

Expand to 5 terms:

16. $(1 + x)^{1/3}$. **17.** $(1 + x)^{1/5}$. **18.** $(1 + x)^{-3/7}$.

Use the binomial expansion to obtain:

19. $\sqrt{40}$. **20.** $\sqrt{60}$. **21.** $\sqrt[3]{30}$.

Find to 3 decimals:

22. $(1.03)^{1/6}$. **23.** $(1.02)^{1/12}$. **24.** $(1.04)^{-1/6}$.

25. Square the expansion of $(1 + x)^{1/2}$ to 5 terms and compare the result with $1 + x$.

26. Show that the first and last terms of (5) suggest that we put

$$_nC_0 = 1, \qquad _nC_n = 1.$$

Evaluate to as many decimals as can readily be done:

27. $\frac{1}{.97}$. **28.** $\frac{1}{1.04}$. **29.** $\frac{1}{96}$. **30.** $\frac{1}{995}$.

31. Show by division that

$$\frac{1}{1 - x} = 1 + x + x^2 + \cdots + x^n + \frac{x^{n+1}}{x - 1}.$$

Discuss the value of the final term when $-1 < x < 1$ in case n is large, and interpret the result carefully.

II. ARITHMETIC PROGRESSIONS

4. Definition. An *arithmetic progression* is a sequence of numbers each of which is obtained from the preceding number by adding a constant called the *common difference.*

Examples. The sequences

$$\begin{gathered} 2, 5, 8, 11, \\ -3, 2, 7, 12, 17, \\ 16, 12, 8, 4, 0, -4, -8, \\ 1/5, 8/15, 13/15, \end{gathered}$$

are arithmetic progressions with differences 3, 5, -4, 1/3, respectively.

In dealing with arithmetic progressions in general we let

a = the first term,
d = the common difference,
l = the last term,
n = the number of terms.

The progression is then

$$a, \quad a + d, \quad a + 2d, \quad \cdots, \quad l - d, \quad l.$$

It is clear that

$$(10) \qquad r\text{th term} = a + (r-1)d.$$

Since the last term is the nth term we have

$$(11) \qquad l = a + (n-1)d.$$

Example 1. How many integers between 100 and 200 are divisible by 7?

The numbers in question form an arithmetic progression with $d = 7$. By trial we find $a = 105$, $l = 196$. Using (11) we have

$$105 + 7(n-1) = 196.$$

Hence

$$7(n-1) = 91, \qquad n - 1 = 13, \qquad n = 14.$$

Example 2. The 7th term of an arithmetic progression is 8, the 21st term is 50. Find the 1st term and the difference.

We have by (10)

$$\begin{aligned} a + 6d &= 8, \\ a + 20d &= 50. \end{aligned}$$

Subtracting,

$$14d = 42.$$

Hence

$$d = 3, \qquad a = 8 - 6d = -10.$$

5. The sum. Let S be the sum of the arithmetic progression. Then

$$S = a + (a+d) + (a+2d) + \cdots + (l-2d) + (l-d) + l.$$

Also

$$S = l + (l-d) + (l-2d) + \cdots + (a+2d) + (a+d) + a.$$

Adding the expressions above, term by term, we have

$$2S = (a+l) + (a+l) + (a+l) + \cdots + (a+l) \quad (n \text{ terms}),$$

or

$$2S = n(a+l),$$

so that

$$(12) \qquad S = \frac{n}{2}(a+l).$$

Substituting (11) in (12) we have

$$(13) \qquad S = \frac{n}{2}[2a + (n-1)d].$$

Example 1. Find the sum of the integers between 100 and 200 divisible by 7.

By example 1, section 4, we know that $a = 105$, $l = 196$, $n = 14$. Hence by (12)

$$S = \frac{14}{2}(105 + 196) = 7(301) = 2107.$$

Example 2. Find the sum of 40 terms of the progression -22, -16, -10, $\cdots$.

Using (13) we have

$$S = \frac{40}{2}(-44 + 39 \times 6) = 20(190) = 3800.$$

EXERCISES

1. Find the terms indicated in the following progressions:

(a) 7th term of $4, 9, 14, \cdots$,
(b) 12th term of $-11, -8, -5, \cdots$,
(c) 25th term of $19, 14, 9, \cdots$,
(d) 10th term of $-2/9, -5/63, 4/63, \cdots$.

2. Find the sums of the progressions in Exercise 1 to the number of terms indicated.

3. Find the sum of the integers between 150 and 525 divisible by 9.

4. Find the sum of the integers between 75 and 400 divisible by 11.

5. The 6th term of an arithmetic progression is 28, the 20th term is 98; find the first term and the difference.

6. Without finding the last terms, obtain the sums of the following progressions to the number of terms indicated.

(a) $2, 8, 14, \cdots$ to 18 terms,
(b) $-5, -1, 3, \cdots$ to 23 terms,
(c) $30, 24, 18, \cdots$ to 40 terms.

7. How many terms of the progression $7, 12, 17, \cdots$ are required to give a sum of 1426? (The solution of a quadratic equation is necessary)

8. How many of the numbers $13, 9, 5, \cdots$ are required to give a sum of -108?

9. Eliminate n between (11) and (13) and find S in terms of a, d, and l.

III. GEOMETRIC PROGRESSIONS

6. Definition. A *geometric progression* is a sequence of numbers each of which is obtained from the preceding number by multiplying by a constant called the *common ratio.*

Examples. The sequences

$$3, 6, 12, 24,$$
$$2, -6, 18, -54, 162,$$
$$4, \frac{4}{3}, \frac{4}{9},$$
$$-2, 1, -\frac{1}{2}, \frac{1}{4},$$

are geometric progressions with ratios 2, -3, $1/3$, $-1/2$, respectively.

In general we let

$a =$ the first term,
$r =$ the common ratio,
$l =$ the last term,
$n =$ the number of terms.

The progression is then

$$a, \quad ar, \quad ar^2, \quad \cdots, \quad l/r, \quad l.$$

It is clear that

$$l = ar^{n-1}. \tag{14}$$

7. The sum. Let S be the sum of the progression. Then

$$S = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}.$$

Hence

$$rS = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n.$$

Subtracting we find

$$S - rS = a - ar^n,$$

that is,

$$S(1 - r) = a(1 - r^n).$$

Therefore

$$S = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}. \tag{15}$$

Example. Find the sum of

$$2 + 6 + 18 + 54 + \cdots$$

to 10 terms.

We have by (15)

$$S = \frac{2(3^{10} - 1)}{3 - 1} = 3^{10} - 1 = 59{,}048.$$

8. Unlimited progression. We can write the value of S in the form

$$S = \frac{a}{1 - r} - \frac{a}{1 - r}r^n. \tag{16}$$

The first term on the right in (16) is independent of the number of terms considered. Now suppose that the ratio r is numerically smaller than 1, that is, that

$$-1 < r < 1.$$

The quantity r^n can then be made as small as desired by taking n sufficiently great. That is, the second term on the right in (16) approaches zero as n increases indefinitely.

We are thus led to the following statement.

If r is numerically less than $1(-1 < r < 1)$ we have

$$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}. \tag{17}$$

It should be observed that the quantity on the right in (17) is not a sum in the literal sense that it is actually attained, but is a limit. The quantity in question is called the *sum to infinity* of the given series.

Example. Find the sum to infinity of

$$6 + 3 + \frac{3}{2} + \cdots.$$

We have by (17)

$$S = \frac{6}{1 - \frac{1}{2}} = 12.$$

It is to be observed that the expansions for $1/(1 + x)$ and $1/(1 - x)$ given in (8) and (9) as obtained by the binomial theorem are given at once by (17).

It should be observed that Exercise 31, page 180 gives a way to obtain (15).

EXERCISES

Find the sums of the following progressions:

1. 3, 6, 12, $\cdots$ to 8 terms.

2. (1.01), $(1.01)^2$, $(1.01)^3$, $\cdots$ to 10 terms, using the binomial theorem to find $(1.01)^{10}$ to 3 decimals.

3. 6, 3, 3/2, $\cdots$ to 6 terms.

4. 6, 3, 3/2, $\cdots$ to infinity.

5. -2, 2/3, $-2/9$, $\cdots$ to infinity.

6. 4, 1.2, .36, $\cdots$ to infinity.

7. -3.4, .68, $-.136$, $\cdots$ to infinity.

IV. LOGARITHMS

9. Definition. The discussion given here is limited to the common logarithms used in computation. The *common logarithm* of a number N is the power to which 10 must be raised to give the number. We have, for example,

$$10^1 = 10, \quad 10^2 = 100, \quad 10^3 = 1000, \quad 10^4 = 10{,}000, \quad \text{etc.,}$$

and we write

$$\log 10 = 1, \quad \log 100 = 2, \quad \log 1000 = 3, \quad \log 10{,}000 = 4, \quad \text{etc.}$$

Further we have

$$10^0 = 1, \quad 10^{-1} = \frac{1}{10} = .1, \quad 10^{-2} = \frac{1}{10^2} = .01, \quad \text{etc.}$$

so that

$$\log 1 = 0, \quad \log .1 = -1, \quad \log .01 = -2, \quad \text{etc.}$$

With the general definition of exponents given in algebra it is to be expected that 37, for example, is a power of 10 between 1 and 2, 249 a power of 10 between 2 and 3, etc. The powers in question are the logarithms of the respective numbers. They cannot be told by inspection, but must be taken from prepared tables.

The meaning of a logarithm should be clearly understood. For example, let us accept the fact that log 37 = 1.568202 to 6 decimals. This means that

$$10^{1.568202} = 37,$$

that is,

$$10^{\frac{1568202}{1000000}} = 37,$$

or

$$10^{1568202} = 37^{1000000}.$$

Here we have ordinary positive integral exponents denoting multiplication. The two numbers indicated would not be exactly equal, owing to the fact that log 37 was used to only 6 decimal places.

10. Laws concerning logarithms. The great aid that logarithms furnish in computation arises from three simple laws, which are direct deductions from the definition of a logarithm and the laws of exponents of algebra.

Let

$$\log M = x, \quad \log N = y;$$

then by the definition of logarithms,

$$M = 10^x, \quad N = 10^y.$$

We now have

$$MN = 10^x \cdot 10^y = 10^{x+y};$$

that is,

$$\log MN = x + y = \log M + \log N.$$

Further

$$\frac{M}{N} = \frac{10^x}{10^y} = 10^{x-y};$$

that is,

$$\log \frac{M}{N} = x - y = \log M - \log N.$$

Finally,

$$M^n = (10)^{nx} = 10^{nx};$$

that is,

$$\log M^n = nx = n \log M.$$

We thus have the laws:

(1) *The logarithm of the product of two numbers is the sum of the logarithms of the numbers. (The extension to the product of more than two numbers is immediate.)*

(2) *The logarithm of a fraction is the logarithm of the numerator minus the logarithm of the denominator.*

(3) *The logarithm of the power of a number is the exponent that denotes the power multiplied by the logarithm of the number.*

From law (1) it is seen that the original problem of calculating the logarithms of integers was reduced to that of obtaining the logarithms of the prime numbers.

EXERCISE

Given log 2 = 0.30 1030, log 3 = 0.47 7121, find log 4, log 6, log 8, log 18, log 24, log 72.

11. Characteristic. The integral part of the logarithm of a number is called the *characteristic* and is not (except in special cases) given in tables of logarithms. It can be easily supplied.

We recall that

$$\log 1 = 0, \quad \log 10 = 1, \quad \log 100 = 2, \quad \log 1000 = 3, \quad \text{etc.}$$

From this it is seen that the characteristic is 0 if there is one figure to the left of the decimal point in the number, the characteristic is 1 if there are 2 figures to the left of the decimal point in the number, etc. Therefore:

If in the number there are figures to the left of the decimal point, the characteristic is positive and is 1 *less than the number of figures to the left of the decimal point.*

Examples. log 7.34 = 0.———, log 18.24 = 1.———,
log 478.24 = 2.———, log 2645 = 3.———,

where the dash represents the decimal portion of the logarithm.

The case of a negative characteristic is best understood by starting with a specific instance. Let us accept the fact that

$$\log 7.34 = 0.86\ 5696.$$

By the second law above we have

$$\begin{aligned}\log .734 &= \log\frac{7.34}{10} = \log 7.34 - \log 10\\ &= 0.86\,5696 - 1 = -.13\,4304.\end{aligned}$$

In practice, the decimal portion of a logarithm is kept positive, so that the final step taken above is not made. Further, negative integers are written as follows:

$$\begin{aligned}-1 &= 9 - 10,\\ -2 &= 8 - 10,\\ -3 &= 7 - 10,\end{aligned}$$

and so on. In the case above we write

$$\log .734 = 0.86\,5696 + 9 - 10 = 9.86\,5696 - 10.$$

Similarly

$$\begin{aligned}\log .0734 &= 0.86\,5696 - 2 = 8.86\,5696 - 10,\\ \log .00734 &= 0.96\,5696 - 3 = 7.86\,5696 - 10,\end{aligned}$$

and so on. The characteristics in the three cases are -1, -2, -3, or $9 - 10$, $8 - 10$, $7 - 10$.

We have the following rule.

If there are no figures to the left of the decimal point in the number, the characteristic is negative and is numerically 1 *larger than the number of zeros between the decimal point and the first significant figure in the number.*

Example. log .93 = 9.——— −10, log .0784 = 8.——— −10,
log. 000 3247 = 6.——— −10.

12. Tables. The decimal part of the logarithm of a number is called the *mantissa*, and it must be taken from a table. In logarithmic tables the decimal point in front of the mantissa is omitted (except in the cases where the characteristic is given). The mantissa depends solely on the sequence of significant figures in the number and not upon the position of the decimal point. For example, the mantissas of the logarithms of

$$4756,\quad 475.6,\quad 47.56,\quad .04756,\quad .004\,756,$$

are the same, the logarithms of these numbers differing only in the characteristics.

In the table in this book, Table XV, the mantissas are given to six figures. In most of the table the mantissas can be taken out without interpolation for four significant figures in the number;

for some of the logarithms in the earlier part of the table five figures in the number are provided for. The first three figures in the number are found in the column headed N, and the fourth figure is found in that one of the ten columns of figures at the top of which the fourth figure of the number appears. All six figures of the mantissa are not printed each time. One goes backwards in the table and picks up the first two figures, unless he finds an asterisk in front of the final four figures of the mantissa, in which case he proceeds forward for the first two figures.

After the mantissa has been taken out the characteristic is supplied in accordance with the proper rule.

Examples.

$\log 34.96 = 1.54\ 3571$, $\log 6762 = 3.83\ 0075$,
$\log 593.2 = 2.77\ 3201$, $\log .007\ 766 = 7.89\ 0197 - 10$.

In the part of the table where five figures in the number are provided for, the procedure is similar.

Examples. $\log 101.24 = 2.00\ 5352$, $\log .10236 = 9.01\ 0130 - 10$.

13. Interpolation. When the number has more significant figures than are provided for directly by the table it is necessary to *interpolate.*

Example. Find log 26.5427.

We have
$$\log 26.5400 = 1.42\ 3901$$
$$\log 26.5500 = 1.42\ 4065.$$

The difference between the two logarithms, namely the *tabular difference,* is 164 in the 6th place. Evidently the *correction* to be added to the smaller logarithm to obtain the required logarithm is, in units of the sixth place,

$$.27 \times 164 = 44.28, \text{ or } 44.$$

Therefore

$$\log 26.5427 = 1.42\ 3945.$$

The calculation of the correction is greatly facilitated by the *proportional parts* that occur in the table. The column on the right of each page, headed *Diff.*, gives the tabular difference across the line. (The difference may not actually be the same all the way across a line, but a glance at the last figures will give the correct difference, though no appreciable error will occur if the printed difference is used.) The proportional parts at the bottom of the page give the various tenths of the differences. By moving the decimal point in the tenths one place to the left, .01, .02, .03, $\cdots$,

.09 of any difference may be found. By adding such a hundredth to the proper tenth, one can obtain any hundredth of a difference. With a little practice the correction for the logarithm of a number of six figures can be obtained and applied mentally. Only in the first portion of the table, where tabular differences are large, should the beginner find it necessary to write down any but the briefest calculation in taking out the logarithm of a number.*

Example. In the case of log 26.5427 above we take the difference 164 in the proportional parts, observe that .2 of it is 32.8, that .07 of it is 11.48, and thus that .27 of it is 44, which is the correction to be added.

EXERCISES

Find the logarithms of the following numbers:

1. 73.463.
2. 296.48.
3. .043 216.
4. 347.237.
5. 654,384.
6. .347 294.
7. 253.2638.
8. 11.36293.
9. 1.024 327.

14. To find a number from its logarithm. When log N is known and N is desired, one proceeds as follows. The characteristic is at first ignored,† for it serves only to place the decimal point properly in the answer. Find the mantissa in the table next smaller than that of log N, obtaining the first 4 figures of N (or in the early part of the table the first 5 figures). Subtract the tabular mantissa from the given one, obtaining the correction. Note the tabular difference and in the proportional parts find a tenth and a hundredth that added will give approximately the correction. In this way the 5th and 6th figures of N are obtained (or the 6th and 7th figures in the early part of the table). Point off the result properly by observing the characteristic.

Example. Given log $N = 2.25\ 3478$, find N.

The next smaller mantissa is .25 3338, corresponding to the figures 1792. Subtracting, we have a difference of 140. The tabular difference is 242. From the proportional parts it is seen that .5 of 242 is 121, and that .08 of 242 is 19.36. Thus .58 of 242 is 140.36, or 140, the difference above. Therefore the 5th figure is 5 and the 6th figure is 8. Consequently $N = 179.258$.

---

* In making logarithmic computations it is convenient to have a slip of paper about 2 inches by 4 inches, to use in following across the lines in the table, and for making calculations incident to interpolation. The final logarithm is all that should go on the page of actual calculation.

Suppose that the correction to be applied to obtain a logarithm is a number such as 17.5. This can be called 17 or 18 with equal propriety. It is customary to take the value so that the 6th figure of the final logarithm is an even, rather than an odd number.

† Sometimes the characteristic should not be ignored at the start. Thus suppose log $N = 7.45\ 2637 - 10$, and N is desired only to 6 decimal places. The characteristic $7 - 10$ shows that there are 2 zeros to the right of the decimal point in N. Thus only 4 figures need be obtained, and no interpolation would be required.

EXERCISES

Find the numbers in the following cases:

1. $\log N = 1.32\ 5496$.
2. $\log N = 2.75\ 2543$.
3. $\log N = 0.45\ 0325$.
4. $\log N = 3.01\ 1156$.
5. $\log N = 9.12\ 2534 - 10$.
6. $\log N = 8.37\ 2562 - 10$.

15. Computations. The laws stated in section 10 govern the combinations of logarithms that occur in making numerical calculations. Mantissas are always kept positive by the proper employment of the $10 - 10$ idea. One manipulates so that the only negative numbers that occur are -10's.

Example 1. Subtract the logarithm 2.34 2516 from the logarithm 1.24 9532.
We add and subtract 10 from the second logarithm, and the subtraction appears as below:

$$\begin{array}{r} 11.24\ 9532 - 10 \\ 2.34\ 2516 \\ \hline 8.90\ 7016 - 10. \end{array}$$

Example 2. Divide the logarithm $9.47\ 2563 - 10$ by 3.
In order to have the negative part of the result -10 we add and subtract 20 from the given logarithm and have the division that follows:

$$\begin{array}{r} 3)29.47\ 2563 - 30 \\ 9.82\ 4188 - 10. \end{array}$$

Example 3. Add the logarithms 3.26 9324 and $8.76\ 2947\ -10$.
The characteristic is changed from $12 - 10$ to 2 as follows:

$$\begin{array}{l} 3.26\ 9324 \\ 8.76\ 2947 - 10 \\ \hline 2.03\ 2271. \end{array}$$

Illustrations of calculations will now be given. The student should study them in detail.

Example 4. Find $N = 7.4263 \times 63.5489$.
The computation follows:

log 7.4263	0 87 0772
log 63.5489	1.80 3108
log N	2.67 3880
N	471.933

Example 5. Find $N = \dfrac{(3.625)^2\sqrt[3]{87.254}}{\sqrt{475.238}}$.

log 3.625	0.55 9308
log 87.254	1.94 0785
log 475.238	2.67 6911
2 log 3.625	1.11 8616
$\frac{1}{3}$ log 87.254	0.64 6928
log "num"	1.76 5544
$\frac{1}{2}$ log 475.238	1.33 8450
log N	0.42 7094
N	2.6736

Example 6. Find $N = \dfrac{246.364 \times 34.769 \times .262}{9234 \times 125.364 \times .084}$.

log 246.364	2.39 1577
log 34.769	1.54 1192
log .262	9.41 8301 − 10
log "num"	13.35 1070 − 10
log "denom"	4.98 7842
log N	8.36 3228 − 10
N	0.023 0796

log 9234	3.96 5390
log 125.364	2.09 8173
log .084	8.92 4279 − 10
log "denom"	4 98 7842

In a logarithmic computation the work should all be outlined, then all logarithms looked up before any additions, subtractions, or other operations are performed.

Experienced computers do not actually write the −10 that occurs when the characteristic is negative, except in the rare cases where confusion would result if it were omitted. An inspection of the last example will show the student that they could have been omitted without misunderstanding.

16. Cologarithms. The quotient of one number by a second number is the same as the product of the first number by the reciprocal of the second number. That is,

$$\frac{M}{N} = M\left(\frac{1}{N}\right),$$

so that

$$\log \frac{M}{N} = \log M + \log \left(\frac{1}{N}\right).$$

The logarithm of the reciprocal of a number is called the *cologarithm* of the number. That is,

$$\operatorname{colog} N = \log \frac{1}{N}.$$

The cologarithm can be obtained very quickly from the logarithm for

$$\operatorname{colog} N = \log 1 - \log N.$$

But

$$\log 1 = 0 = 10 - 10,$$

and this enables one to write colog N at once. We have in fact the rule.

To find the cologarithm of a number, subtract every figure in the logarithm of the number from 9, *except the last figure* (*or the last non-zero figure*); *subtract it from* 10. *Add* − 10, *unless the logarithm of the number included* −10.

Examples. If log N = 1.36 2462, then colog N = 8.63 7538 − 10
log N = 2.47 3600, then colog N = 7.52 6400 − 10
log N = 8.25 8736 − 10, then colog N = 1.74 1264.

With a little practice the cologarithm of a number can be written as readily as the logarithm after looking up the latter in the table, unless an extensive interpolation is needed.

When cologarithms are employed it is not necessary to combine separately the logarithms of factors in the numerator and in the denominator, and a single addition can replace two additions and a subtraction. Cologarithms are systematically used by experienced computers.

Example. Find $N = \frac{243.628 \times 2.936}{3.698 \times 72.134}$.

log 243.628	2.38 6727
log 2.936	0.46 7756
colog 3.698	9.43 2033 − 10
colog 72.134	8.14 1860 − 10
log N	0.42 8376
N	2.68 149

EXERCISES

A

Make the calculations indicated by means of logarithms. In general, 6 significant figures should be given in the result.*

1. 63.25×246.34.

2. $\dfrac{375 \times 2896}{47.84 \times 6.98}$.

3. $\dfrac{.476\,245 \times .038\,694 \times 7.364\,250}{.007\,264}$.

4. $\dfrac{\sqrt{793.465} \times \sqrt[3]{36.264}}{37.291}$.

5. $\dfrac{(36.25)^2\sqrt{496.37}}{(3.48)^3\sqrt[3]{27.95}}$.

6. $\dfrac{3.26475 \times \sqrt[3]{8.15296}}{(4.75312)^2 \times \sqrt{2.11315}}$.

7. $8325(1.0452)^7$.

8. $3000(1.0325)^{-10}$.

9. $245.38(1\,025)^{5/12}$.

10. $3250.88(1.0475)^{-3/4}$.

11. $2500\,\dfrac{(1.0325)^{12} - 1}{.0325}$.

12. $350\,\dfrac{1 - (1.045)^{-8}}{.045}$.

13. $12[(1.05)^{1/12} - 1]$.

14. $\dfrac{.04}{6[(1.04)^{1/6} - 1]}$.

B

1. 9.286×24.796.

2. $\dfrac{832 \times 1211}{96.14 \times 7.93}$.

3. $\dfrac{.236740 \times 6.248280 \times .007865}{.072964 \times .378246}$.

4. $\dfrac{\sqrt{24.396} \times \sqrt[3]{425.211}}{(36.24)^2}$.

5. $\dfrac{(8.93)^2\sqrt{47.236}}{(35.12)^3\sqrt[3]{.1468}}$.

6. $2634(1.0315)^6$.

7. $456.50(1.0412)^{-8}$.

8. $945(1.063)^{7/12}$.

9. $6212.74(1.0375)^{-5/6}$.

10. $3200\,\dfrac{(1\,0425)^{15} - 1}{.0425}$.

11. $485.25\,\dfrac{1 - (1.06)^{-9}}{.06}$.

12. $\dfrac{.0325}{4[(1.0325)^{1/4} - 1]}$.

---

* The 6th significant figure is not always correct when 6-place logarithms are used, even though there has been no error in the work. In Example 5 above, only 5 figures were given in the answer because in dividing by 2 a choice of two possible 6-place logarithms had to be made, and one could be quite certain that the 6th figure would not be accurate.

TABLES

The tables of investment functions that follow are taken with permission from James W. Glover's *Tables of Applied Mathematics*, published by George Wahr, Ann Arbor, Michigan The tables of Glover give a wide range of interest rates, provide for many conversion periods, and are to more places of decimals than the abridgment here given They are designed to meet the needs of extensive practice.

Table I—The Number of Each Day of the Year

Day of Month	Jan.	Feb.	Mar.	April	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.	Day of Month
1	1	32	60	91	121	152	182	213	244	274	305	335	**1**
2	2	33	61	92	122	153	183	214	245	275	306	336	**2**
3	3	34	62	93	123	154	184	215	246	276	307	337	**3**
4	4	35	63	94	124	155	185	216	247	277	308	338	**4**
5	5	36	64	95	125	156	186	217	248	278	309	339	**5**
6	6	37	65	96	126	157	187	218	249	279	310	340	**6**
7	7	38	66	97	127	158	188	219	250	280	311	341	**7**
8	8	39	67	98	128	159	189	220	251	281	312	342	**8**
9	9	40	68	99	129	160	190	221	252	282	313	343	**9**
10	10	41	69	100	130	161	191	222	253	283	314	344	**10**
11	11	42	70	101	131	162	192	223	254	284	315	345	**11**
12	12	43	71	102	132	163	193	224	255	285	316	346	**12**
13	13	44	72	103	133	164	194	225	256	286	317	347	**13**
14	14	45	73	104	134	165	195	226	257	287	318	348	**14**
15	15	46	74	105	135	166	196	227	258	288	319	349	**15**
16	16	47	75	106	136	167	197	228	259	289	320	350	**16**
17	17	48	76	107	137	168	198	229	260	290	321	351	**17**
18	18	49	77	108	138	169	199	230	261	291	322	352	**18**
19	19	50	78	109	139	170	200	231	262	292	323	353	**19**
20	20	51	79	110	140	171	201	232	263	293	324	354	**20**
21	21	52	80	111	141	172	202	233	264	294	325	355	**21**
22	22	53	81	112	142	173	203	234	265	295	326	356	**22**
23	23	54	82	113	143	174	204	235	266	296	327	357	**23**
24	24	55	83	114	144	175	205	236	267	297	328	358	**24**
25	25	56	84	115	145	176	206	237	268	298	329	359	**25**
26	26	57	85	116	146	177	207	238	269	299	330	360	**26**
27	27	58	86	117	147	178	208	239	270	300	331	361	**27**
28	28	59	87	118	148	179	209	240	271	301	332	362	**28**
29	29		88	119	149	180	210	241	272	302	333	363	**29**
30	30		89	120	150	181	211	242	273	303	334	364	**30**
31	31		90		151		212	243		304		365	**31**

Note.—For leap years the number of the day is one greater than the number in the table after February 28.

Table II—Ordinary and Exact Simple Interest on 1000 at 1%

Days	Ordinary Interest	Exact Interest	Days	Ordinary Interest	Exact Interest
1	0.027 7778	0.027 3973	150	4.166 6667	4 109 5890
2	0.055 5556	0.054 7945	160	4.444 4444	4 383 5616
3	0.083 3333	0.082 1918	170	4.722 2222	4 657 5342
4	0.111 1111	0.109 5890	180	5.000 0000	4.931 5068
5	0.138 8889	0.136 9863	190	5.277 7778	5 205 4795
6	0.166 6667	0.164 3836	200	5.555 5556	5 479 4521
7	0.194 4444	0.191 7808	210	5.833 3333	5.753 4247
8	0.222 2222	0.219 1781	220	6.111 1111	6.027 3973
9	0.250 0000	0.246 5753	230	6.388 8889	6 301 3699
10	0.277 7778	0.273 9726	240	6.666 6667	6 575 3425
20	0.555 5556	0.547 9452	250	6.944 4444	6 849 3151
30	0.833 3333	0.821 9178	260	7.222 2222	7.123 2877
40	1.111 1111	1.095 8904	270	7.500 0000	7 397 2603
50	1.388 8889	1.369 8630	280	7.777 7778	7.671 2329
60	1.666 6667	1.643 8356	290	8.055 5556	7.945 2055
70	1.944 4444	1.917 8082	300	8.333 3333	8.219 1781
80	2.222 2222	2.191 7808	310	8.611 1111	8.493 1507
90	2.500 0000	2.465 7534	320	8.888 8889	8.767 1233
100	2.777 7778	2.739 7260	330	9.166 6667	9.041 0959
110	3.055 5556	3.013 6986	340	9.444 4444	9.315 0685
120	3.333 3333	3.287 6712	350	9.722 2222	9.589 0411
130	3.611 1111	3.561 6438	360	10.000 0000	9.863 0137
140	3.888 8889	3.835 6164			

Table III—Amount of 1 at Compound Interest

$(1 + i)^n$

n	½%	1%	1¼%	1½%	n
1	1.005 0000	1.010 0000	1.012 5000	1 015 0000	1
2	1.010 0250	1.020 1000	1.025 1562	1.030 2250	2
3	1.015 0751	1.030 3010	1.037 9707	1.045 6784	3
4	1 020 1505	1.040 6040	1.050 9453	1.061 3636	4
5	1.025 2513	1.051 0101	1.064 0822	1.077 2840	5
6	1.030 3775	1 061 5202	1.077 3832	1.093 4433	6
7	1.035 5294	1.072 1354	1.090 8505	1.109 8449	7
8	1.040 7070	1.082 8567	1.104 4861	1.126 4926	8
9	1.045 9106	1.093 6853	1.118 2922	1.143 3900	9
10	1.051 1401	1.104 6221	1.132 2708	1.160 5408	10
11	1.056 3958	1.115 6683	1.146 4242	1.177 9489	11
12	1.061 6778	1.126 8250	1.160 7545	1.195 6182	12
13	1.066 9862	1.138 0933	1.175 2639	1.213 5524	13
14	1.072 3211	1.149 4742	1.189 9547	1.231 7557	14
15	1.077 6827	1.160 9690	1.204 8292	1.250 2321	15
16	1.083 0712	1.172 5786	1.219 8895	1.268 9855	16
17	1.088 4865	1.184 3044	1.235 1382	1.288 0203	17
18	1.093 9289	1.196 1475	1.250 5774	1.307 3406	18
19	1.099 3986	1.208 1090	1.266 2096	1.326 9507	19
20	1.104 8956	1.220 1900	1.282 0372	1.346 8550	20
21	1.110 4201	1.232 3919	1.298 0627	1.367 0578	21
22	1.115 9722	1.244 7159	1.314 2885	1.387 5637	22
23	1.121 5520	1.257 1630	1.330 7171	1.408 3772	23
24	1.127 1598	1.269 7346	1.347 3511	1.429 5028	24
25	1.132 7956	1.282 4320	1.364 1929	1.450 9454	25
26	1.138 4596	1.295 2563	1.381 2454	1.472 7095	26
27	1.144 1519	1.308 2089	1.398 5109	1.494 8002	27
28	1.149 8726	1.321 2910	1.415 9923	1.517 2222	28
29	1 155 6220	1.334 5039	1.433 6922	1.539 9805	29
30	1.161 4001	1.347 8489	1.451 6134	1.563 0802	30
31	1.167 2071	1.361 3274	1.469 7585	1.586 5264	31
32	1.173 0431	1.374 9407	1.488 1305	1.610 3243	32
33	1.178 9083	1.388 6901	1.506 7321	1.634 4792	33
34	1.184 8029	1.402 5770	1.525 5663	1.658 9964	34
35	1.190 7269	1.416 6028	1.544 6359	1.683 8813	35
36	1.196 6805	1.430 7688	1.563 9438	1.709 1395	36
37	1.202 6639	1.445 0765	1.583 4931	1.734 7766	37
38	1.208 6772	1.459 5272	1.603 2868	1.760 7983	38
39	1.214 7206	1.474 1225	1.623 3279	1.787 2103	39
40	1.220 7942	1.488 8637	1.643 6195	1.814 0184	40
41	1.226 8982	1.503 7524	1.664 1647	1.841 2287	41
42	1.233 0327	1.518 7899	1.684 9668	1.868 8471	42
43	1.239 1979	1.533 9778	1.706 0289	1.896 8798	43
44	1.245 3939	1.549 3176	1.727 3542	1.925 3330	44
45	1.251 6208	1.564 8107	1.748 9461	1.954 2130	45
46	1.257 8789	1.580 4589	1.770 8080	1.983 5262	46
47	1.264 1683	1.596 2634	1.792 9431	2.013 2791	47
48	1.270 4892	1.612 2261	1.815 3548	2.043 4783	48
49	1.276 8416	1.628 3483	1.838 0468	2.074 1305	49
50	1.283 2258	1.644 6318	1.861 0224	2.105 2424	50
60	1.348 8502	1.816 6967	2.107 1814	2.443 2198	60
70	1.417 8305	2.006 7634	2.385 9000	2 835 4563	70
80	1.490 3386	2.216 7152	2.701 4849	3 290 6628	80
90	1.566 5547	2.448 6327	3.058 8126	3.818 9485	90
100	1.646 6685	2.704 8138	3.463 4043	4.432 0456	100

Table III—Logarithm of Amount of 1 at Compound Interest

$\log (1 + i)^n$

n	½%	1%	1¼%	1½%	n
1	0.00 2166	0.00 4321	0.00 5395	0.00 6466	1
2	0.00 4332	0.00 8643	0.01 0790	0.01 2932	2
3	0.00 6498	0.01 2964	0.01 6185	0.01 9398	3
4	0.00 8664	0.01 7285	0.02 1580	0.02 5864	4
5	0.01 0830	0.02 1607	0.02 6975	0.03 2330	5
6	0.01 2996	0.02 5928	0.03 2370	0.03 8796	6
7	0.01 5162	0.03 0250	0.03 7765	0.04 5262	7
8	0.01 7328	0.03 4571	0.04 3160	0.05 1728	8
9	0.01 9495	0.03 8892	0.04 8555	0.05 8194	9
10	0.02 1661	0.04 3214	0.05 3950	0.06 4660	10
11	0.02 3827	0.04 7535	0.05 9345	0.07 1126	11
12	0.02 5993	0.05 1856	0.06 4740	0.07 7593	12
13	0.02 8159	0.05 6178	0.07 0135	0.08 4059	13
14	0.03 0325	0.06 0499	0.07 5530	0.09 0525	14
15	0.03 2491	0.06 4821	0.08 0925	0.09 6991	15
16	0.03 4657	0.06 9142	0.08 6321	0.10 3457	16
17	0.03 6823	0.07 3463	0.09 1716	0.10 9923	17
18	0.03 8989	0.07 7785	0.09 7111	0.11 6389	18
19	0.04 1155	0.08 2106	0.10 2506	0.12 2855	19
20	0.04 3321	0.08 6427	0.10 7901	0.12 9321	20
21	0.04 5487	0.09 0749	0.11 3296	0.13 5787	21
22	0.04 7653	0.09 5070	0.11 8691	0.14 2253	22
23	0.04 9819	0.09 9392	0.12 4086	0.14 8719	23
24	0.05 1985	0.10 3713	0.12 9481	0.15 5185	24
25	0.05 4152	0.10 8034	0.13 4876	0.16 1651	25
26	0.05 6318	0.11 2356	0.14 0271	0.16 8117	26
27	0.05 8484	0.11 6677	0.14 5666	0.17 4583	27
28	0.06 0650	0.12 0998	0.15 1061	0.18 1049	28
29	0.06 2816	0.12 5320	0.15 6456	0.18 7515	29
30	0.06 4982	0.12 9641	0.16 1851	0.19 3981	30
31	0.06 7148	0.13 3963	0.16 7246	0.20 0447	31
32	0.06 9314	0.13 8284	0.17 2641	0.20 6913	32
33	0.07 1480	0.14 2605	0.17 8036	0.21 3379	33
34	0.07 3646	0.14 6927	0.18 3431	0.21 9845	34
35	0.07 5812	0.15 1248	0.18 8826	0.22 6311	35
36	0.07 7978	0.15 5569	0.19 4221	0.23 2778	36
37	0.08 0144	0.15 9891	0.19 9616	0.23 9244	37
38	0.08 2310	0.16 4212	0.20 5011	0.24 5710	38
39	0.08 4476	0.16 8534	0.21 0406	0.25 2176	39
40	0.08 6642	0.17 2855	0.21 5801	0.25 8642	40
41	0.08 8809	0.17 7176	0.22 1196	0.26 5108	41
42	0.09 0975	0.18 1498	0.22 6591	0.27 1574	42
43	0.09 3141	0.18 5819	0.23 1986	0.27 8040	43
44	0.09 5307	0.19 0140	0.23 7381	0.28 4506	44
45	0.09 7473	0.19 4462	0.24 2776	0.29 0972	45
46	0.09 9639	0.19 8783	0.24 8171	0.29 7438	46
47	0.10 1805	0.20 3105	0.25 3566	0.30 3904	47
48	0.10 3971	0.20 7426	0.25 8962	0.31 0370	48
49	0.10 6137	0.21 1747	0.26 4357	0.31 6836	49
50	0.10 8303	0.21 6069	0.26 9752	0.32 3302	50
60	0.12 9964	0.25 9282	0.32 3702	0.38 7963	60
70	0.15 1624	0.30 2496	0.37 7652	0.45 2623	70
80	0.17 3285	0.34 5710	0.43 1603	0.51 7283	80
90	0.19 4946	0.38 8924	0.48 5553	0.58 1944	90
100	0.21 6606	0.43 2137	0.53 9503	0.64 6604	100

,Table III—Amount of 1 at Compound Interest

$(1 + i)^n$

n	$1\frac{3}{4}\%$	2%	$2\frac{1}{2}\%$	3%	n
1	1.017 5000	1.020 0000	1.025 0000	1.030 0000	1
2	1.035 3062	1.040 4000	1.050 6250	1.060 9000	2
3	1.053 4241	1.061 2080	1.076 8906	1.092 7270	3
4	1.071 8590	1.082 4322	1.103 8129	1.125 5088	4
5	1.090 6166	1.104 0808	1.131 4082	1.159 2741	5
6	1.109 7024	1.126 1624	1.159 6934	1.194 0523	6
7	1.129 1221	1.148 6857	1.188 6858	1.229 8739	7
8	1.148 8818	1.171 6594	1.218 4029	1.266 7701	8
9	1.168 9872	1.195 0926	1.248 8630	1.304 7732	9
10	1.189 4445	1.218 9944	1.280 0845	1.343 9164	10
11	1.210 2598	1.243 3743	1.312 0867	1.384 2339	11
12	1.231 4393	1.268 2418	1.344 8888	1.425 7609	12
13	1.252 9895	1.293 6066	1.378 5110	1.468 5337	13
14	1.274 9168	1.319 4788	1.412 9738	1.512 5897	14
15	1.297 2279	1.345 8683	1.448 2982	1.557 9674	15
16	1.319 9294	1.372 7857	1.484 5056	1.604 7064	16
17	1.343 0281	1.400 2414	1.521 6183	1.652 8476	17
18	1.366 5311	1.428 2462	1.559 6587	1.702 4331	18
19	1.390 4454	1.456 8112	1.598 6502	1.753 5061	19
20	1.414 7782	1.485 9474	1.638 6164	1.806 1112	20
21	1.439 5368	1.515 6663	1.679 5819	1.860 2946	21
22	1.464 7287	1.545 9797	1.721 5714	1 916 1034	22
23	1.490 3615	1.576 8993	1.764 6107	1.973 5865	23
24	1.516 4428	1.608 4372	1.808 7259	2.032 7941	24
25	1.542 9805	1.640 6060	1.853 9441	2.093 7779	25
26	1.569 9827	1.673 4181	1.900 2927	2.156 5913	26
27	1.597 4574	1.706 8865	1.947 8000	2.221 2890	27
28	1.625 4129	1.741 0242	1.996 4950	2.287 9277	28
29	1.653 8576	1.775 8447	2.046 4074	2.356 5655	29
30	1.682 8001	1.811 3616	2.097 5676	2.427 2625	30
31	1.712 2491	1.847 5888	2.150 0068	2.500 0803	31
32	1.742 2135	1.884 5406	2.203 7569	2.575 0828	32
33	1.772 7022	1.922 2314	2.258 8509	2.652 3352	33
34	1.803 7245	1.960 6760	2.315 3221	2.731 9053	34
35	1.835 2897	1.999 8896	2.373 2052	2.813 8624	35
36	1.867 4073	2.039 8873	2.432 5353	2.898 2783	36
37	1.900 0869	2.080 6851	2.493 3487	2.985 2267	37
38	1.933 3384	2.122 2988	2.555 6824	3.074 7835	38
39	1.967 1718	2.164 7448	2.619 5745	3.167 0270	39
40	2.001 5973	2.208 0397	2.685 0638	3.262 0378	40
41	2.036 6253	2.252 2005	2.752 1904	3.359 8989	41
42	2.072 2662	2.297 2445	2.820 9952	3.460 6959	42
43	2.108 5309	2.343 1894	2.891 5201	3.564 5168	43
44	2.145 4302	2.390 0531	2.963 8081	3.671 4523	44
45	2.182 9752	2.437 8542	3.037 9033	3.781 5958	45
46	2.221 1773	2.486 6113	3.113 8509	3.895 0437	46
47	2.260 0479	2.536 3435	3.191 6971	4.011 8950	47
48	2.299 5987	2.587 0704	3.271 4896	4.132 2519	48
49	2.339 8417	2.638 8118	3.353 2768	4.256 2194	49
50	2.380 7889	2.691 5880	3.437 1087	4.383 9060	50
60	2.831 8163	3.281 0308	4.399 7897	5.891 6031	60
70	3.368 2883	3.999 5582	5.632 1029	7.917 8219	70
80	4.006 3919	4.875 4392	7.209 5678	10.640 8906	80
90	4.765 3808	5.943 1331	9.228 8563	14.300 4671	90
100	5.668 1559	7.244 6461	11.813 7164	19.218 6320	100

Table III—Logarithm of Amount of 1 at Compound Interest

$\log (1 + i)^n$

n	1¾%	2%	2½%	3%	n
1	0.00 7534	0.00 8600	0.01 0724	0.01 2837	1
2	0.01 5069	0.01 7200	0.02 1448	0.02 5674	2
3	0.02 2603	0.02 5801	0.03 2172	0.03 8512	3
4	0.03 0138	0.03 4401	0.04 2895	0.05 1349	4
5	0.03 7672	0.04 3001	0.05 3619	0.06 4186	5
6	0.04 5207	0.05 1601	0.06 4343	0.07 7023	6
7	0.05 2741	0.06 0201	0.07 5067	0.08 9861	7
8	0.06 0275	0.06 8801	0.08 5791	0.10 2698	8
9	0.06 7810	0.07 7402	0.09 6515	0.11 5535	9
10	0.07 5344	0.08 6002	0.10 7239	0.12 8372	10
11	0.08 2879	0.09 4602	0.11 7963	0.14 1209	11
12	0.09 0413	0.10 3202	0.12 8686	0.15 4047	12
13	0.09 7947	0.11 1802	0.13 9410	0.16 6884	13
14	0.10 5482	0.12 0402	0.15 0134	0.17 9721	14
15	0.11 3016	0.12 9003	0.16 0858	0.19 2558	15
16	0.12 0551	0.13 7603	0.17 1582	0.20 5396	16
17	0.12 8085	0.14 6203	0.18 2306	0.21 8233	17
18	0.13 5620	0.15 4803	0.19 3030	0.23 1070	18
19	0.14 3154	0.16 3403	0.20 3753	0.24 3907	19
20	0.15 0688	0.17 2003	0.21 4477	0.25 6744	20
21	0.15 8223	0.18 0604	0.22 5201	0.26 9582	21
22	0.16 5757	0.18 9204	0.23 5925	0.28 2419	22
23	0.17 3292	0.19 7804	0.24 6649	0.29 5256	23
24	0.18 0826	0.20 6404	0.25 7373	0.30 8093	24
25	0.18 8360	0.21 5004	0.26 8097	0.32 0931	25
26	0.19 5895	0.22 3604	0.27 8820	0.33 3768	26
27	0.20 3429	0.23 2205	0.28 9544	0.34 6605	27
28	0.21 0964	0.24 0805	0.30 0268	0.35 9442	28
29	0.21 8498	0.24 9405	0.31 0992	0.37 2280	29
30	0.22 6033	0.25 8005	0.32 1716	0.38 5117	30
31	0.23 3567	0.26 6605	0.33 2440	0.39 7954	31
32	0.24 1101	0.27 5205	0.34 3164	0.41 0791	32
33	0.24 8636	0.28 3806	0.35 3888	0.42 3628	33
34	0.25 6170	0.29 2406	0.36 4611	0.43 6466	34
35	0.26 3705	0.30 1006	0.37 5335	0.44 9303	35
36	0.27 1239	0.30 9606	0.38 6059	0.46 2140	36
37	0.27 8773	0.31 8206	0.39 6783	0.47 4977	37
38	0.28 6308	0.32 6807	0.40 7507	0.48 7815	38
39	0.29 3842	0.33 5407	0.41 8231	0.50 0652	39
40	0.30 1377	0.34 4007	0.42 8955	0.51 3489	40
41	0.30 8911	0.35 2607	0.43 9678	0.52 6326	41
42	0.31 6446	0.36 1207	0.45 0402	0.53 9163	42
43	0.32 3980	0.36 9807	0.46 1126	0.55 2001	43
44	0.33 1514	0.37 8408	0.47 1850	0.56 4838	44
45	0.33 9049	0.38 7008	0.48 2574	0.57 7675	45
46	0.34 6583	0.39 5608	0.49 3298	0.59 0512	46
47	0.35 4118	0.40 4208	0.50 4022	0.60 3350	47
48	0.36 1652	0.41 2808	0.51 4746	0.61 6187	48
49	0.36 9186	0.42 1408	0.52 5469	0.62 9024	49
50	0.37 6721	0.43 0009	0.53 6193	0.64 1861	50
60	0.45 2065	0.51 6010	0.64 3432	0.77 0234	60
70	0.52 7409	0.60 2012	0.75 0671	0.89 8606	70
80	0.60 2753	0.68 8014	0.85 7909	1.02 6978	80
90	0.67 8098	0.77 4015	0.96 5148	1.15 5350	90
100	0.75 3442	0.86 0017	1.07 2387	1.28 3723	100

Table III—Amount of 1 at Compound Interest

$(1 + i)^n$

n	3½%	4%	4½%	5%	n
1	1.035 0000	1.040 0000	1.045 0000	1.050 0000	1
2	1.071 2250	1.081 6000	1.092 0250	1.102 5000	2
3	1.108 7179	1.124 8640	1.141 1661	1.157 6250	3
4	1.147 5230	1.169 8586	1.192 5186	1.215 5062	4
5	1.187 6863	1.216 6529	1.246 1819	1.276 2816	5
6	1.229 2553	1.265 3190	1.302 2601	1.340 0956	6
7	1.272 2793	1.315 9318	1.360 8618	1.407 1004	7
8	1.316 8090	1.368 5691	1.422 1006	1.477 4554	8
9	1.362 8974	1.423 3118	1.486 0951	1.551 3282	9
10	1.410 5988	1.480 2443	1.552 9694	1.628 8946	10
11	1.459 9697	1.539 4541	1.622 8530	1.710 3394	11
12	1.511 0687	1.601 0322	1.695 8814	1.795 8563	12
13	1.563 9561	1.665 0735	1.772 1961	1.885 6491	13
14	1.618 6945	1.731 6764	1.851 9449	1.979 9316	14
15	1.675 3488	1.800 9435	1.935 2824	2.078 9282	15
16	1.733 9860	1.872 9812	2.022 3702	2.182 8746	16
17	1.794 6756	1.947 9005	2.113 3768	2.292 0183	17
18	1.857 4892	2.025 8165	2.208 4788	2.406 6192	18
19	1.922 5013	2.106 8492	2.307 8603	2.526 9502	19
20	1.989 7889	2.191 1231	2.411 7140	2.653 2977	20
21	2.059 4315	2.278 7681	2.520 2412	2.785 9626	21
22	2.131 5116	2.369 9188	2.633 6520	2.925 2607	22
23	2.206 1145	2.464 7155	2.752 1663	3.071 5238	23
24	2.283 3285	2.563 3042	2.876 0138	3.225 0999	24
25	2.363 2450	2.665 8363	3.005 4345	3.386 3549	25
26	2.445 9586	2.772 4698	3.140 6790	3.555 6727	26
27	2.531 5671	2.883 3686	3.282 0096	3.733 4563	27
28	2.620 1720	2.998 7033	3.429 7000	3.920 1291	28
29	2.711 8780	3.118 6515	3.584 0365	4.116 1356	29
30	2.806 7937	3.243 3975	3.745 3181	4.321 9424	30
31	2.905 0315	3.373 1334	3.913 8575	4.538 0395	31
32	3.006 7076	3.508 0587	4.089 9810	4.764 9415	32
33	3.111 9424	3.648 3811	4.274 0302	5.003 1885	33
34	3.220 8603	3.794 3163	4.466 3615	5.253 3480	34
35	3.333 5904	3.946 0890	4.667 3478	5.516 0154	35
36	3.450 2661	4.103 9326	4.877 3785	5.791 8161	36
37	3.571 0254	4.268 0899	5.096 8605	6.081 4069	37
38	3.696 0113	4.438 8135	5.326 2192	6.385 4773	38
39	3.825 3717	4.616 3660	5.565 8991	6.704 7512	39
40	3.959 2597	4.801 0206	5.816 3645	7.039 9887	40
41	4.097 8338	4.993 0615	6.078 1009	7.391 9882	41
42	4.241 2580	5.192 7839	6.351 6155	7.761 5876	42
43	4.389 7020	5.400 4953	6.637 4382	8.149 6669	43
44	4.543 3416	5.616 5151	6.936 1229	8.557 1503	44
45	4.702 3586	5.841 1757	7.248 2484	8.985 0078	45
46	4.866 9411	6.074 8227	7.574 4196	9.434 2582	46
47	5.037 2840	6.317 8156	7.915 2685	9.905 9711	47
48	5.213 5890	6.570 5282	8.271 4556	10.401 2696	48
49	5.396 0646	6.833 3494	8.643 6711	10.921 3331	49
50	5.584 9269	7.106 6833	9.032 6363	11.467 3998	50
60	7.878 0909	10.519 6274	14.027 4079	18.679 1859	60
70	11.112 8253	15.571 6184	21.784 1356	30.426 4255	70
80	15.675 7375	23.049 7991	33.830 0964	49.561 4411	80
90	22.112 1759	34.119 3333	52.537 1053	80.730 3650	90
100	31.191 4080	50.504 9482	81.588 5180	131.501 2578	100

Table III—Logarithm of Amount of 1 at Compound Interest

$\log (1 + i)^n$

n	3½%	4%	4½%	5%	n
1	0 01 4940	0.01 7033	0.01 9116	0.02 1189	1
2	0 02 9881	0.03 4067	0.03 8233	0.04 2379	2
3	0.04 4821	0.05 1100	0.05 7349	0.06 3568	3
4	0.05 9761	0.06 8133	0 07 6465	0.08 4757	4
5	0.07 4702	0.08 5167	0.09 5581	0 10 5946	5
6	0.08 9642	0.10 2200	0.11 4698	0.12 7136	6
7	0.10 4582	0.11 9233	0.13 3814	0.14 8325	7
8	0.11 9523	0.13 6267	0.15 2930	0.16 9514	8
9	0 13 4463	0.15 3300	0.17 2047	0 19 0704	9
10	0.14 9403	0.17 0333	0.19 1163	0.21 1893	10
11	0.16 4344	0.18 7367	0.21 0279	0.23 3082	11
12	0.17 9284	0.20 4400	0.22 9395	0 25 4272	12
13	0.19 4225	0.22 1433	0.24 8512	0.27 5461	13
14	0.20 9165	0.23 8467	0.26 7628	0.29 6650	14
15	0.22 4105	0.25 5500	0.28 6744	0.31 7839	15
16	0.23 9046	0.27 2533	0.30 5861	0.33 9029	16
17	0.25 3986	0.28 9567	0.32 4977	0.36 0218	17
18	0.26 8926	0.30 6600	0.34 4093	0.38 1407	18
19	0.28 3867	0.32 3633	0 36 3210	0.40 2597	19
20	0.29 8807	0.34 0667	0.38 2326	0.42 3786	20
21	0.31 3747	0.35 7700	0.40 1442	0.44 4975	21
22	0.32 8688	0.37 4733	0.42 0558	0.46 6165	22
23	0.34 3628	0.39 1767	0 43 9675	0.48 7354	23
24	0.35 8568	0.40 8800	0 45 8791	0.50 8543	24
25	0.37 3509	0.42 5833	0.47 7907	0.52 9732	25
26	0.38 8449	0.44 2867	0.49 7024	0 55 0922	26
27	0.40 3389	0.45 9900	0.51 6140	0 57 2111	27
28	0.41 8330	0 47 6933	0.53 5256	0 59 3300	28
29	0.43 3270	0 49 3967	0.55 4372	0.61 4490	29
30	0.44 8211	0.51 1000	0.57 3489	0.63 5679	30
31	0.46 3151	0.52 8034	0.59 2605	0.65 6868	31
32	0.47 8091	0.54 5067	0.61 1721	0.67 8058	32
33	0.49 3032	0.56 2100	0 63 0838	0.69 9247	33
34	0.50 7972	0.57 9134	0 64 9954	0.72 0436	34
35	0.52 2912	0.59 6167	0.66 9070	0.74 1625	35
36	0.53 7853	0.61 3200	0.68 8186	0.76 2815	36
37	0.55 2793	0.63 0234	0.70 7303	0.78 4004	37
38	0.56 7733	0.64 7267	0.72 6419	0.80 5193	38
39	0.58 2674	0.66 4300	0.74 5535	0.82 6383	39
40	0.59 7614	0.68 1334	0.76 4652	0.84 7572	40
41	0.61 2554	0.69 8367	0.78 3768	0.86 8761	41
42	0.62 7495	0.71 5400	0 80 2884	0.88 9951	42
43	0.64 2435	0.73 2434	0.82 2000	0.91 1140	43
44	0.65 7375	0.74 9467	0 84 1117	0.93 2329	44
45	0.67 2316	0.76 6500	0.86 0233	0.95 3518	45
46	0.68 7256	0.78 3534	0.87 9349	0.97 4708	46
47	0.70 2196	0.80 0567	0.89 8466	0.99 5897	47
48	0.71 7137	0.81 7600	0.91 7582	1 01 7086	48
49	0.73 2077	0 83 4634	0.93 6698	1.03 8276	49
50	0.74 7018	0 85 1667	0.95 5815	1.05 9465	50
60	0.89 6421	1.02 2000	1.14 6977	1.27 1358	60
70	1.04 5824	1.19 2334	1.33 8140	1.48 3251	70
80	1.19 5228	1 36 2667	1 52 9303	1.69 5144	80
90	1.34 4631	1.53 3000	1.72 0466	1.90 7037	90
100	1.49 4035	1.70 3334	1.91 1629	2.11 8930	100

Table III—Amount of 1 at Compound Interest

$$(1+i)^n$$

n	5½%	6%	7%	8%	n
1	1.055 0000	1.060 0000	1.070 0000	1.080 0000	**1**
2	1.113 0250	1.123 6000	1.144 9000	1.166 4000	**2**
3	1.174 2414	1.191 0160	1.225 0430	1.259 7120	**3**
4	1.238 8247	1.262 4770	1.310 7960	1.360 4890	**4**
5	1.306 9600	1.338 2256	1.402 5517	1.469 3281	**5**
6	1.378 8428	1.418 5191	1.500 7304	1.586 8743	**6**
7	1.454 6792	1.503 6303	1.605 7815	1.713 8243	**7**
8	1.534 6865	1.593 8481	1.718 1862	1.850 9302	**8**
9	1.619 0943	1.689 4790	1.838 4592	1.999 0046	**9**
10	1.708 1445	1.790 8477	1.967 1514	2.158 9250	**10**
11	1.802 0924	1.898 2986	2.104 8520	2.331 6390	**11**
12	1.901 2075	2.012 1965	2.252 1916	2.518 1701	**12**
13	2.005 7739	2.132 9283	2.409 8450	2.719 6237	**13**
14	2.116 0915	2.260 9040	2.578 5342	2 937 1936	**14**
15	2.232 4765	2.396 5582	2.759 0315	3.172 1691	**15**
16	2.355 2627	2.540 3517	2.952 1637	3.425 9426	**16**
17	2.484 8021	2.692 7728	3.158 8152	3.700 0181	**17**
18	2.621 4663	2.854 3392	3.379 9323	3.996 0195	**18**
19	2.765 6469	3.025 5995	3.616 5275	4.315 7011	**19**
20	2.917 7575	3.207 1355	3.869 6845	4.660 9571	**20**
21	3.078 2342	3.399 5636	4.140 5624	5.033 8337	**21**
22	3.247 5370	3.603 5374	4.430 4017	5.436 5404	**22**
23	3.426 1516	3.819 7497	4.740 5299	5.871 4636	**23**
24	3.614 5899	4.048 9346	5.072 3670	6.341 1807	**24**
25	3.813 3923	4.291 8707	5.427 4326	6.848 4752	**25**
26	4 023 1289	4.549 3830	5.807 3529	7.396 3532	**26**
27	4.244 4010	4.822 3459	6.213 8676	7.988 0615	**27**
28	4 477 8431	5.111 6867	6 648 8384	8.627 1064	**28**
29	4.724 1244	5.418 3879	7 114 2570	9.317 2749	**29**
30	4.983 9513	5.743 4912	7.612 2550	10.062 6569	**30**
31	5.258 0686	6.088 1006	8.145 1129	10.867 6694	**31**
32	5.547 2624	6.453 3867	8.715 2708	11.737 0830	**32**
33	5.852 3618	6.840 5899	9.325 3398	12.676 0496	**33**
34	6.174 2417	7.251 0253	9.978 1135	13.690 1336	**34**
35	6.513 8250	7.686 0868	10.676 5815	14.785 3443	**35**
36	6.872 0854	8.147 2520	11.423 9422	15.968 1718	**36**
37	7.250 0501	8 636 0871	12.223 6181	17.245 6256	**37**
38	7.648 8028	9.154 2523	13.079 2714	18.625 2756	**38**
39	8.069 4870	9.703 5075	13.994 8204	20.115 2977	**39**
40	8.513 3088	10.285 7179	14.974 4578	21.724 5215	**40**
41	8.981 5408	10.902 8610	16.022 6699	23.462 4832	**41**
42	9.475 5255	11.557 0327	17.144 2568	25.339 4819	**42**
43	9 996 6794	12.250 4546	18.344 3548	27.366 6404	**43**
44	10.546 4968	12.985 4819	19.628 4596	29.555 9717	**44**
45	11.126 5541	13.764 6108	21.002 4518	31.920 4494	**45**
46	11.738 5146	14.590 4875	22.472 6234	34.474 0853	**46**
47	12.384 1329	15.465 9167	24 045 7070	37.232 0122	**47**
48	13 065 2602	16.393 8717	25.728 9065	40.210 5731	**48**
49	13.783 8495	17.377 5040	27 529 9300	43.427 4190	**49**
50	14.541 9612	18.420 1543	29.547 0251	46.901 6125	**50**
60	24.839 7704	32.987 6909	57.946 4268	101.257 0637	**60**
70	42.429 9162	59.075 9302	113.989 3922	218.606 4059	**70**
80	72.476 4263	105.795 9935	224.234 3876	471 954 8343	**80**
90	123.800 2059	189.464 5112	441.102 9799	1018.915 0893	**90**
100	211.468 6357	339 302 0835	867.716 3256	2199.761 2563	**100**

Table III—Logarithm of Amount of 1 at Compound Interest

$\log\,(1 + i)^n$

n	5½%	6%	7%	8%	n
1	0.02 3252	0.02 5306	0.02 9384	0.03 3424	1
2	0.04 6505	0.05 0612	0.05 8768	0.06 6848	2
3	0.06 9757	0.07 5918	0.08 8151	0.10 0271	3
4	0.09 3010	0.10 1223	0.11 7535	0.13 3695	4
5	0.11 6262	0.12 6529	0.14 6919	0.16 7119	5
6	0.13 9515	0.15 1835	0.17 6303	0.20 0543	6
7	0.16 2767	0.17 7141	0.20 5686	0.23 3966	7
8	0.18 6020	0.20 2447	0.23 5070	0.26 7390	8
9	0.20 9272	0.22 7753	0.26 4454	0.30 0814	9
10	0.23 2525	0.25 3059	0.29 3838	0.33 4238	10
11	0.25 5777	0.27 8365	0.32 3222	0.36 7661	11
12	0.27 9030	0.30 3670	0.35 2605	0.40 1085	12
13	0.30 2282	0.32 8976	0.38 1989	0.43 4509	13
14	0.32 5534	0.35 4282	0.41 1373	0.46 7933	14
15	0.34 8787	0.37 9588	0.44 0757	0.50 1356	15
16	0.37 2039	0.40 4894	0.47 0140	0.53 4780	16
17	0.39 5292	0.43 0200	0.49 9524	0.56 8204	17
18	0.41 8544	0.45 5506	0.52 8908	0.60 1628	18
19	0.44 1797	0.48 0811	0.55 8292	0.63 5051	19
20	0.46 5049	0.50 6117	0.58 7676	0.66 8475	20
21	0.48 8302	0.53 1423	0.61 7059	0.70 1899	21
22	0.51 1554	0.55 6729	0.64 6443	0.73 5323	22
23	0.53 4807	0.58 2035	0.67 5827	0.76 8746	23
24	0.55 8059	0.60 7341	0.70 5211	0.80 2170	24
25	0.58 1312	0.63 2647	0.73 4594	0.83 5594	25
26	0.60 4564	0.65 7952	0.76 3978	0.86 9018	26
27	0.62 7816	0.68 3258	0.79 3362	0.90 2441	27
28	0.65 1069	0.70 8564	0.82 2746	0.93 5865	28
29	0.67 4321	0.73 3870	0.85 2130	0.96 9289	29
30	0.69 7574	0.75 9176	0.88 1513	1.00 2713	30
31	0.72 0826	0.78 4482	0.91 0897	1.03 6136	31
32	0.74 4079	0.80 9788	0.94 0281	1.06 9560	32
33	0.76 7331	0.83 5094	0.96 9665	1.10 2984	33
34	0.79 0584	0.86 0399	0.99 9048	1.13 6408	34
35	0.81 3836	0.88 5705	1.02 8432	1.16 9831	35
36	0.83 7089	0.91 1011	1.05 7816	1.20 3255	36
37	0.86 0341	0.93 6317	1.08 7200	1.23 6679	37
38	0.88 3593	0.96 1623	1.11 6584	1.27 0103	38
39	0.90 6846	0.98 6929	1.14 5967	1.30 3526	39
40	0.93 0098	1.01 2235	1.17 5351	1.33 6950	40
41	0.95 3351	1.03 7540	1.20 4735	1.37 0374	41
42	0.97 6603	1.06 2846	1.23 4119	1.40 3798	42
43	0.99 9856	1.08 8152	1.26 3502	1.43 7222	43
44	1.02 3108	1.11 3458	1.29 2886	1.47 0645	44
45	1.04 6361	1.13 8764	1.32 2270	1.50 4069	45
46	1.06 9613	1.16 4070	1.35 1654	1.53 7493	46
47	1.09 2866	1.18 9376	1.38 1038	1.57 0917	47
48	1.11 6118	1.21 4682	1.41 0421	1.60 4340	48
49	1.13 9371	1.23 9987	1.43 9805	1.63 7764	49
50	1.16 2623	1.26 5293	1.46 9189	1.67 1188	50
60	1.39 5148	1.51 8352	1.76 3027	2.00 5425	60
70	1.62 7672	1.77 1411	2.05 6864	2.33 9663	70
80	1.86 0197	2.02 4469	2.35 0702	2.67 3900	80
90	2.09 2721	2.27 7528	2.64 4540	3.00 8138	90
100	2.32 5246	2.53 0587	2.93 8378	3.34 2376	100

Table IV—Present Value of 1 at Compound Interest

$$v^n = (1 + i)^{-n}$$

n	½%	1%	1¼%	1½%	n
1	0.995 0249	0.990 0990	0.987 6543	0.985 2217	1
2	0.990 0745	0.980 2960	0.975 4611	0.970 6617	2
3	0.985 1488	0.970 5901	0.963 4183	0.956 3170	3
4	0.980 2475	0.960 9803	0.951 5243	0.942 1842	4
5	0.975 3707	0.951 4657	0.939 7771	0.928 2603	5
6	0.970 5181	0.942 0452	0.928 1749	0.914 5422	6
7	0.965 6896	0.932 7181	0.916 7159	0.901 0268	7
8	0.960 8852	0.923 4832	0.905 3984	0.887 7111	8
9	0.956 1047	0.914 3398	0.894 2207	0.874 5022	9
10	0.951 3479	0.905 2870	0.883 1809	0.861 6672	10
11	0.946 6149	0.896 3237	0.872 2775	0.848 9332	11
12	0.941 9053	0.887 4492	0.861 5086	0.836 3874	12
13	0.937 2192	0.878 6626	0.850 8727	0.824 0270	13
14	0.932 5565	0.869 9630	0.840 3681	0.811 8493	14
15	0.927 9169	0.861 3495	0.829 9932	0.799 8515	15
16	0.923 3004	0.852 8213	0.819 7463	0.788 0310	16
17	0.918 7068	0.844 3775	0.809 6260	0.776 3853	17
18	0.914 1362	0.836 0173	0.799 6306	0.764 9116	18
19	0.909 5882	0.827 7399	0.789 7587	0.753 6075	19
20	0.905 0629	0.819 5445	0.780 0085	0.742 4704	20
21	0.900 5601	0.811 4302	0.770 3788	0.731 4979	21
22	0.896 0797	0.803 3962	0.760 8680	0.720 6876	22
23	0.891 6216	0.795 4418	0.751 4745	0.710 0371	23
24	0.887 1857	0.787 5661	0.742 1971	0.699 5439	24
25	0.882 7718	0.779 7684	0.733 0341	0.689 2058	25
26	0.878 3799	0.772 0480	0.723 9843	0.679 0205	26
27	0.874 0099	0.764 4039	0.715 0463	0.668 9857	27
28	0.869 6616	0.756 8356	0.706 2185	0.659 0992	28
29	0.865 3349	0.749 3421	0.697 4998	0.649 3589	29
30	0.861 0297	0.741 9229	0.688 8887	0.639 7624	30
31	0.856 7460	0.734 5771	0.680 3839	0.630 3078	31
32	0.852 4836	0.727 3041	0.671 9841	0.620 9929	32
33	0.848 2424	0.720 1031	0.663 6880	0.611 8157	33
34	0.844 0223	0.712 9733	0.655 4943	0.602 7741	34
35	0.839 8231	0.705 9142	0.647 4018	0.593 8661	35
36	0.835 6449	0.698 9249	0.639 4092	0.585 0897	36
37	0.831 4875	0.692 0049	0.631 5152	0.576 4431	37
38	0.827 3507	0.685 1534	0.623 7187	0.567 9242	38
39	0.823 2346	0.678 3697	0.616 0185	0.559 5313	39
40	0.819 1389	0.671 6531	0.608 4133	0.551 2623	40
41	0.815 0635	0.665 0031	0.600 9021	0.543 1156	41
42	0.811 0085	0.658 4189	0.593 4835	0.535 0892	42
43	0.806 9736	0.651 8999	0.586 1566	0.527 1815	43
44	0.802 9588	0.645 4455	0.578 9201	0.519 3907	44
45	0.798 9640	0.639 0549	0.571 7729	0.511 7149	45
46	0.794 9891	0.632 7276	0.564 7140	0.504 1527	46
47	0.791 0339	0.626 4630	0.557 7422	0.496 7021	47
48	0.787 0984	0.620 2604	0.550 8565	0.489 3617	48
49	0.783 1825	0.614 1192	0.544 0558	0.482 1297	49
50	0.779 2861	0.608 0388	0.537 3391	0.475 0047	50
60	0.741 3722	0.550 4496	0.474 5676	0.409 2960	60
70	0.705 3029	0.498 3149	0.419 1291	0.352 6769	70
80	0.670 9885	0.451 1179	0.370 1668	0.303 8901	80
90	0.638 3435	0.408 3912	0.326 9242	0.261 8522	90
100	0.607 2868	0.369 7112	0.288 7333	0.225 6294	100

Table IV—Present Value of 1 at Compound Interest

$$v^n = (1 + i)^{-n}$$

n	$1\frac{3}{4}\%$	2%	$2\frac{1}{2}\%$	3%	n
1	0.982 8010	0.980 3922	0.975 6098	0.970 8738	**1**
2	0.965 8978	0.961 1688	0.951 8144	0.942 5959	**2**
3	0.949 2853	0.942 3223	0.928 5994	0.915 1417	**3**
4	0.932 9585	0.923 8454	0.905 9506	0.888 4870	**4**
5	0.916 9125	0.905 7308	0.883 8543	0.862 6088	**5**
6	0.901 1425	0.887 9714	0.862 2969	0.837 4843	**6**
7	0.885 6438	0.870 5602	0.841 2652	0.813 0915	**7**
8	0.870 4116	0.853 4904	0.820 7466	0.789 4092	**8**
9	0.855 4414	0.836 7553	0.800 7284	0.766 4167	**9**
10	0.840 7286	0.820 3483	0.781 1984	0.744 0939	**10**
11	0.826 2689	0.804 2630	0.762 1448	0.722 4213	**11**
12	0.812 0579	0.788 4932	0.743 5559	0.701 3799	**12**
13	0.798 0913	0.773 0325	0.725 4204	0.680 9513	**13**
14	0.784 3649	0.757 8750	0.707 7272	0.661 1178	**14**
15	0.770 8746	0.743 0147	0.690 4656	0.641 8619	**15**
16	0.757 6163	0.728 4458	0.673 6249	0.623 1669	**16**
17	0.744 5861	0.714 1626	0.657 1951	0.605 0164	**17**
18	0.731 7799	0.700 1594	0.641 1659	0.587 3946	**18**
19	0.719 1940	0.686 4308	0.625 5277	0.570 2860	**19**
20	0.706 8246	0.672 9713	0.610 2709	0.553 6758	**20**
21	0.694 6679	0.659 7758	0.595 3863	0.537 5493	**21**
22	0.682 7203	0.646 8390	0.580 8647	0.521 8925	**22**
23	0.670 9782	0.634 1559	0.566 6972	0.506 6917	**23**
24	0.659 4380	0.621 7215	0.552 8754	0.491 9337	**24**
25	0.648 0963	0.609 5309	0.539 3906	0.477 6056	**25**
26	0.636 9497	0.597 5793	0.526 2347	0.463 6947	**26**
27	0.625 9948	0.585 8620	0.513 3997	0.450 1891	**27**
28	0.615 2283	0.574 3746	0.500 8778	0.437 0768	**28**
29	0.604 6470	0.563 1123	0.488 6613	0.424 3464	**29**
30	0.594 2476	0.552 0709	0.476 7427	0.411 9868	**30**
31	0.584 0272	0.541 2460	0.465 1148	0.399 9871	**31**
32	0.573 9825	0.530 6333	0.453 7706	0.388 3370	**32**
33	0.564 1105	0.520 2287	0.442 7030	0.377 0262	**33**
34	0.554 4084	0.510 0282	0.431 9053	0.366 0449	**34**
35	0.544 8731	0.500 0276	0.421 3711	0.355 3834	**35**
36	0.535 5018	0.490 2232	0.411 0937	0.345 0324	**36**
37	0.526 2917	0.480 6109	0.401 0670	0.334 9829	**37**
38	0.517 2400	0.471 1872	0.391 2849	0.325 2262	**38**
39	0.508 3440	0.461 9482	0.381 7414	0.315 7535	**39**
40	0.499 6010	0.452 8904	0.372 4306	0.306 5568	**40**
41	0.491 0083	0.444 0102	0.363 3469	0.297 6280	**41**
42	0.482 5635	0.435 3041	0.354 4848	0.288 9592	**42**
43	0.474 2639	0.426 7688	0.345 8389	0.280 5429	**43**
44	0.466 1070	0.418 4007	0.337 4038	0.272 3718	**44**
45	0.458 0904	0.410 1968	0.329 1744	0.264 4386	**45**
46	0.450 2117	0.402 1537	0.321 1458	0.256 7365	**46**
47	0.442 4685	0.394 2684	0.313 3129	0.249 2588	**47**
48	0.434 8585	0.386 5376	0.305 6712	0.241 9988	**48**
49	0.427 3793	0.378 9584	0.298 2158	0.234 9503	**49**
50	0.420 0288	0.371 5279	0.290 9422	0.228 1071	**50**
60	0.353 1303	0.304 7823	0.227 2836	0.169 7331	**60**
70	0.296 8867	0.250 0276	0.177 5536	0.126 2974	**70**
80	0.249 6011	0.205 1097	0.138 7046	0.093 9771	**80**
90	0.209 8468	0.168 2614	0.108 3558	0.069 9278	**90**
100	0.176 4242	0.138 0330	0.084 6474	0.052 0328	**100**

Table IV—Present Value of 1 at Compound Interest

$$v^n = (1+i)^{-n}$$

n	3½%	4%	4½%	5%	n
1	0.966 1836	0.961 5385	0.956 9378	0.952 3810	1
2	0.933 5107	0.924 5562	0.915 7300	0.907 0295	2
3	0.901 9427	0.888 9964	0.876 2966	0.863 8376	3
4	0.871 4422	0.854 8042	0.838 5613	0.822 7025	4
5	0.841 9732	0.821 9271	0.802 4510	0.783 5262	5
6	0.813 5006	0.790 3145	0.767 8957	0.746 2154	6
7	0.785 9910	0.759 9178	0.734 8285	0.710 6813	7
8	0.759 4116	0.730 6902	0.703 1851	0.676 8394	8
9	0.733 7310	0.702 5867	0.672 9044	0.644 6089	9
10	0.708 9188	0.675 5642	0.643 9277	0.613 9133	10
11	0.684 9457	0.649 5809	0.616 1987	0.584 6793	11
12	0.661 7833	0.624 5970	0.589 6639	0.556 8374	12
13	0.639 4042	0.600 5741	0.564 2716	0.530 3214	13
14	0.617 7818	0.577 4751	0.539 9729	0.505 0680	14
15	0.596 8906	0.555 2645	0.516 7204	0.481 0171	15
16	0.576 7059	0.533 9082	0.494 4693	0.458 1115	16
17	0.557 2038	0.513 3732	0.473 1764	0.436 2967	17
18	0.538 3611	0.493 6281	0.452 8004	0.415 5207	18
19	0.520 1557	0.474 6424	0.433 3018	0.395 7340	19
20	0.502 5659	0.456 3869	0.414 6429	0.376 8895	20
21	0.485 5709	0.438 8336	0.396 7874	0.358 9424	21
22	0.469 1506	0.421 9554	0.379 7009	0.341 8499	22
23	0.453 2856	0.405 7263	0.363 3501	0.325 5713	23
24	0.437 9571	0.390 1215	0.347 7035	0.310 0679	24
25	0.423 1470	0.375 1168	0.332 7306	0.295 3028	25
26	0.408 8377	0.360 6892	0.318 4025	0.281 2407	26
27	0.395 0122	0.346 8166	0.304 6914	0.267 8483	27
28	0.381 6543	0.333 4775	0.291 5707	0.255 0936	28
29	0.368 7482	0.320 6514	0.279 0150	0.242 9463	29
30	0.356 2784	0.308 3187	0.267 0000	0.231 3774	30
31	0.344 2303	0.296 4603	0.255 5024	0.220 3595	31
32	0.332 5897	0.285 0579	0.244 4999	0.209 8662	32
33	0.321 3427	0.274 0942	0.233 9712	0.199 8725	33
34	0.310 4761	0.263 5521	0.223 8959	0.190 3548	34
35	0.299 9769	0.253 4155	0.214 2544	0.181 2903	35
36	0.289 8327	0.243 6687	0.205 0282	0.172 6574	36
37	0.280 0316	0.234 2968	0.196 1992	0.164 4356	37
38	0.270 5619	0.225 2854	0.187 7504	0.156 6054	38
39	0.261 4125	0.216 6206	0.179 6655	0.149 1480	39
40	0.252 5725	0.208 2890	0.171 9287	0.142 0457	40
41	0.244 0314	0.200 2779	0.164 5251	0.135 2816	41
42	0.235 7791	0.192 5749	0.157 4403	0.128 8396	42
43	0.227 8059	0.185 1682	0.150 6605	0.122 7044	43
44	0.220 1023	0.178 0463	0.144 1728	0.116 8613	44
45	0.212 6592	0.171 1984	0.137 9644	0.111 2965	45
46	0.205 4679	0.164 6139	0.132 0233	0.105 9967	46
47	0.198 5197	0.158 2826	0.126 3381	0.100 9492	47
48	0.191 8065	0.152 1948	0.120 8977	0.096 1421	48
49	0.185 3202	0.146 3411	0.115 6916	0.091 5639	49
50	0.179 0534	0.140 7126	0.110 7096	0.087 2037	50
60	0.126 9343	0.095 0604	0.071 2890	0.053 5355	60
70	0.089 9861	0.064 2194	0.045 9050	0.032 8662	70
80	0.063 7929	0.043 3843	0.029 5595	0.020 1770	80
90	0.045 2240	0.029 3089	0.019 0342	0.012 3869	90
100	0.032 0601	0.019 8000	0.012 2566	0.007 6045	100

Table IV—Present Value of 1 at Compound Interest

$$v^n = (1 + i)^{-n}$$

n	$5\frac{1}{2}\%$	6%	7%	8%	n
1	0.947 8673	0.943 3962	0.934 5794	0.925 9259	1
2	0.898 4524	0.889 9964	0.873 4387	0.857 3388	2
3	0.851 6137	0.839 6193	0.816 2979	0.793 8322	3
4	0.807 2167	0.792 0937	0.762 8952	0.735 0299	4
5	0.765 1344	0.747 2582	0.712 9862	0.680 5832	5
6	0.725 2458	0.704 9605	0.666 3422	0.630 1696	6
7	0.687 4368	0.665 0571	0.622 7497	0.583 4904	7
8	0.651 5989	0.627 4124	0.582 0091	0.540 2689	8
9	0.617 6293	0.591 8985	0.543 9337	0.500 2490	9
10	0.585 4306	0.558 3948	0.508 3493	0.463 1935	10
11	0.554 9105	0.526 7875	0.475 0928	0.428 8829	11
12	0.525 9815	0.496 9694	0.444 0120	0.397 1138	12
13	0.498 5607	0.468 8390	0.414 9644	0.367 6979	13
14	0.472 5694	0.442 3010	0.387 8172	0.340 4610	14
15	0.447 9330	0.417 2651	0.362 4460	0.315 2417	15
16	0.424 5811	0.393 6463	0.338 7346	0.291 8905	16
17	0.402 4465	0.371 3644	0.316 5744	0.270 2690	17
18	0.381 4659	0.350 3438	0.295 8639	0.250 2490	18
19	0.361 5791	0.330 5130	0.276 5083	0.231 7121	19
20	0.342 7290	0.311 8047	0.258 4190	0.214 5482	20
21	0.324 8616	0.294 1554	0.241 5131	0.198 6557	21
22	0.307 9257	0.277 5051	0.225 7132	0.183 9405	22
23	0.291 8727	0.261 7973	0.210 9469	0.170 3153	23
24	0.276 6566	0.246 9785	0.197 1466	0.157 6993	24
25	0.262 2337	0.232 9986	0.184 2492	0.146 0179	25
26	0.248 5628	0.219 8100	0.172 1955	0.135 2018	26
27	0.235 6045	0.207 3680	0.160 9304	0.125 1868	27
28	0.223 3218	0.195 6301	0.150 4022	0.115 9137	28
29	0.211 6794	0.184 5567	0.140 5628	0.107 3275	29
30	0.200 6440	0.174 1101	0.131 3671	0.099 3773	30
31	0.190 1839	0.164 2548	0.122 7730	0.092 0160	31
32	0.180 2691	0.154 9574	0.114 7411	0.085 2000	32
33	0.170 8712	0.146 1862	0.107 2347	0.078 8889	33
34	0.161 9632	0.137 9115	0.100 2193	0.073 0453	34
35	0.153 5196	0.130 1052	0.093 6629	0.067 6345	35
36	0.145 5162	0.122 7408	0.087 5355	0.062 6246	36
37	0.137 9301	0.115 7932	0.081 8088	0.057 9857	37
38	0.130 7394	0.109 2389	0.076 4569	0.053 6905	38
39	0.123 9236	0.103 0555	0.071 4550	0.049 7134	39
40	0.117 4631	0.097 2222	0.066 7804	0.046 0309	40
41	0.111 3395	0.091 7190	0.062 4116	0.042 6212	41
42	0.105 5350	0.086 5274	0.058 3286	0.039 4641	42
43	0.100 0332	0.081 6296	0.054 5127	0.036 5408	43
44	0.094 8182	0.077 0091	0.050 9464	0.033 8341	44
45	0.089 8751	0.072 6501	0.047 6135	0.031 3279	45
46	0.085 1897	0.068 5378	0.044 4986	0.029 0073	46
47	0.080 7485	0.064 6583	0.041 5875	0.026 8586	47
48	0.076 5389	0.060 9984	0.038 8668	0.024 8691	48
49	0.072 5487	0.057 5457	0.036 3241	0.023 0269	49
50	0.068 7665	0.054 2884	0.033 9478	0.021 3212	50
60	0.040 2580	0.030 3143	0.017 2573	0.009 8759	60
70	0.023 5683	0.016 9274	0.008 7727	0.004 5744	70
80	0.013 7976	0.009 4522	0.004 4596	0.002 1188	80
90	0.008 0775	0.005 2780	0.002 2670	0.000 9814	90
100	0.004 7288	0.002 9472	0.001 1525	0.000 4546	100

Table V—Amount of 1 per Annum at Compound Interest

$$s_{\overline{n|}} = [(1 + i)^n - 1]/i$$

n	½%	1%	1¼%	1½%	*n*
1	1.000 0000	1 000 0000	1.000 0000	1.000 0000	1
2	2.005 0000	2.010 0000	2.012 5000	2.015 0000	2
3	3.015 0250	3.030 1000	3 037 6562	3.045 2250	3
4	4.030 1001	4 060 4010	4.075 6270	4.090 9034	4
5	5 050 2506	5.101 0050	5.126 5723	5.152 2669	5
6	6 075 5019	6 152 0151	6 190 6544	6.229 5509	6
7	7.105 8794	7.213 5352	7.268 0376	7.322 9942	7
8	8.141 4088	8.285 6706	8.358 8881	8.432 8391	8
9	9 182 1158	9.368 5273	9.463 3742	9.559 3317	9
10	10.228 0264	10.462 2125	10.581 6664	10.702 7217	10
11	11.279 1665	11.566 8347	11.713 9372	11.863 2625	11
12	12.335 5624	12.682 5030	12.860 3614	13 041 2114	12
13	13 397 2402	13 809 3280	14 021 1159	14.236 8296	13
14	14 464 2264	14.947 4213	15 196 3799	15.450 3820	14
15	15 536 5475	16.096 8955	16 386 3346	16.682 1378	15
16	16 614 2303	17.257 8645	17.591 1638	17.932 3698	16
17	17 697 3014	18.430 4431	18 811 0534	19.201 3554	17
18	18.785 7879	19.614 7476	20.046 1915	20.489 3757	18
19	19 879 7169	20.810 8950	21.296 7689	21.796 7164	19
20	20 979 1154	22.019 0040	22.562 9785	23.123 6671	20
21	22 084 0110	23.239 1940	23.845 0158	24.470 5221	21
22	23 194 4311	24.471 5860	25.143 0785	25 837 5799	22
23	24 310 4032	25.716 3018	26.457 3669	27.225 1436	23
24	25.431 9552	26.973 4649	27.788 0840	28.633 5208	24
25	26.559 1150	28.243 1995	29.135 4351	30.063 0236	25
26	27.691 9106	29.525 6315	30.499 6280	31.513 9690	26
27	28.830 3701	30.820 8878	31.880 8734	32 986 6785	27
28	29.974 5220	32.129 0967	33.279 3843	34.481 4787	28
29	31.124 3946	33.450 3877	34.695 3766	35.998 7009	29
30	32.280 0166	34.784 8915	36.129 0688	37.538 6814	30
31	33.441 4167	36.132 7404	37.580 6822	39.101 7616	31
32	34.608 6237	37.494 0679	39.050 4407	40.688 2880	32
33	35.781 6669	38.869 0085	40.538 5712	42.298 6123	33
34	36.960 5752	40.257 6986	42.045 3033	43.933 0915	34
35	38.145 3781	41.660 2756	43.570 8696	45.592 0879	35
36	39.336 1050	43.076 8784	45.115 5055	47.275 9692	36
37	40.532 7855	44.507 6471	46.679 4493	48.985 1087	37
38	41.735 4494	45.952 7236	48.262 9424	50.719 8854	38
39	42.944 1267	47.412 2509	49.866 2292	52.480 6837	39
40	44.158 8473	48.886 3734	51.489 5571	54.267 8939	40
41	45.379 6415	50.375 2371	53.133 1765	56.081 9123	41
42	46.606 5397	51.878 9895	54.797 3412	57.923 1410	42
43	47.839 5724	53.397 7794	56.482 3080	59 791 9881	43
44	49.078 7703	54.931 7572	58.188 3369	61.688 8679	44
45	50.324 1642	56.481 0747	59.915 6911	63.614 2010	45
46	51.575 7850	58.045 8855	61.664 6372	65.568 4140	46
47	52.833 6639	59.626 3443	63.435 4452	67.551 9402	47
48	54 097 8322	61.222 6078	65.228 3882	69.565 2193	48
49	55.368 3214	62.834 8338	67.043 7431	71.608 6976	49
50	56.645 1630	64.463 1822	68.881 7899	73.682 8280	50
60	69.770 0305	81.669 6699	88.574 5078	96.214 6517	60
70	83.566 1055	100.676 3368	110.871 9978	122.363 7529	70
80	98.067 7136	121.671 5217	136.118 7953	152.710 8525	80
90	113 310 9358	144 863 2675	164.705 0076	187.929 9004	90
100	129.333 6984	170.481 3829	197.072 3420	228.803 0433	100

Table V—Logarithm of Amount of 1 per Annum

log $s_{\overline{n}|}$

n	½%	1%	1¼%	1½%	n
1	0.00 0000	0.00 0000	0.00 0000	0.00 0000	1
2	0.30 2114	0.30 3196	0.30 3736	0.30 4275	2
3	0.47 9291	0.48 1457	0.48 2539	0.48 3619	3
4	0.60 5316	0.60 8569	0.61 0194	0.61 1819	4
5	0.70 3313	0.70 7656	0.70 9827	0.71 1998	5
6	0.78 3582	0.78 9017	0.79 1737	0.79 4457	6
7	0.85 1618	0.85 8148	0.86 1417	0.86 4689	7
8	0.91 0700	0.91 8328	0.92 2148	0.92 5974	8
9	0.96 2943	0.97 1671	0.97 6046	0.98 0428	9
10	1.00 9792	1.01 9623	1.02 4554	1.02 9494	10
11	1.05 2277	1.06 3215	1.06 8703	1.07 4204	11
12	1.09 1159	1.10 3205	1.10 9253	1.11 5318	12
13	1.12 7015	1.14 0173	1.14 6783	1.15 3413	13
14	1.16 0295	1.17 4566	1.18 1740	1.18 8939	14
15	1.19 1355	1.20 6742	1.21 4482	1.22 2252	15
16	1.22 0480	1.23 6987	1.24 5294	1.25 3638	16
17	1.24 7907	1.26 5536	1.27 4413	1.28 3332	17
18	1.27 3829	1.29 2583	1.30 2032	1.31 1529	18
19	1.29 8410	1.31 8291	1.32 8314	1.33 8391	19
20	1.32 1787	1.34 2798	1.35 3396	1.36 4057	20
21	1.34 4078	1.36 6221	1.37 7398	1.38 8643	21
22	1.36 5384	1.38 8662	1.40 0418	1.41 2252	22
23	1.38 5792	1.41 0208	1.42 2547	1.43 4970	23
24	1.40 5380	1.43 0937	1.44 3859	1.45 6875	24
25	1.42 4214	1.45 0914	1.46 4421	1.47 8033	25
26	1.44 2353	1.47 0199	1.48 4295	1.49 8503	26
27	1.45 9850	1.48 8845	1.50 3530	1.51 8339	27
28	1.47 6752	1.50 6899	1.52 2175	1.53 7586	28
29	1.49 3110	1.52 4401	1.54 0272	1.55 6287	29
30	1.50 8934	1.54 1391	1.55 7857	1.57 4479	30
31	1.52 4285	1.55 7901	1.57 4965	1.59 2196	31
32	1.53 9184	1.57 3963	1.59 1626	1.60 9469	32
33	1.55 3661	1.58 9603	1.60 7868	1.62 6326	33
34	1.56 7739	1.60 4849	1.62 3717	1.64 2792	34
35	1.58 1442	1.61 9722	1.63 9196	1.65 8889	35
36	1.59 4791	1.63 4244	1.65 4326	1.67 4640	36
37	1.60 7806	1.64 8435	1.66 9126	1.69 0064	37
38	1.62 0505	1.66 2311	1.68 3614	1.70 5178	38
39	1.63 2904	1.67 5891	1.69 7807	1.71 9999	39
40	1.64 5018	1.68 9188	1.71 1719	1.73 4543	40
41	1.65 6861	1.70 2217	1.72 5366	1.74 8823	41
42	1.66 8447	1.71 4992	1.73 8759	1.76 2852	42
43	1.67 9787	1.72 7523	1.75 1912	1.77 6643	43
44	1.69 0894	1.73 9823	1.76 4836	1.79 0207	44
45	1.70 1777	1.75 1903	1.77 7541	1.80 3554	45
46	1.71 2446	1.76 3771	1.79 0036	1.81 6695	46
47	1.72 2911	1.77 5438	1.80 2332	1.82 9638	47
48	1.73 3180	1.78 6912	1.81 4437	1.84 2392	48
49	1.74 3261	1.79 8200	1.82 6358	1.85 4966	49
50	1.75 3163	1.80 9312	1.83 8104	1.86 7366	50
60	1.84 3669	1.91 2061	1.94 7309	1.98 3241	60
70	1.92 2030	2.00 2927	2.04 4822	2.08 7653	70
80	1.99 1526	2.08 5189	2.13 3918	2.18 3870	80
90	2.05 4272	2.16 0958	2.21 6707	2.27 3996	90
100	2.11 1712	2.28 1677	2.29 4626	2.35 9462	100

Table V—Amount of 1 per Annum at Compound Interest

$$s_{\overline{n}|} = [(1 + i)^n - 1]/i$$

n	$1\frac{3}{4}\%$	2%	$2\frac{1}{2}\%$	3%	n
1	1 000 0000	1.000 0000	1 000 0000	1.000 0000	1
2	2 017 5000	2.020 0000	2.025 0000	2.030 0000	2
3	3 052 8062	3.060 4000	3.075 6250	3.090 9000	3
4	4 106 2304	4.121 6080	4.152 5156	4.183 6270	4
5	5.178 0894	5.204 0402	5.256 3285	5.309 1358	5
6	6.268 7060	6.308 1210	6.387 7367	6.468 4099	6
7	7.378 4083	7.434 2834	7.547 4301	7.662 4622	7
8	8.507 5305	8.582 9691	8.736 1159	8.892 3360	8
9	9.656 4122	9.754 6284	9 954 5188	10.159 1061	9
10	10.825 3995	10.949 7210	11.203 3818	11.463 8793	10
11	12.014 8439	12.168 7154	12 483 4663	12.807 7957	11
12	13.225 1037	13.412 0897	13.795 5530	14.192 0296	12
13	14.456 5430	14.680 3315	15.140 4418	15.617 7904	13
14	15.709 5325	15.973 9382	16.518 9528	17.086 3242	14
15	16.984 4493	17.293 4169	17.931 9267	18.598 9139	15
16	18.281 6772	18.639 2853	19.380 2248	20.156 8813	16
17	19.601 6066	20.012 0710	20.864 7304	21.761 5877	17
18	20.944 6347	21.412 3124	22.386 3487	23.414 4354	18
19	22.311 1658	22 840 5586	23.946 0074	25.116 8684	19
20	23.701 6112	24.297 3698	25.544 6576	26.870 3745	20
21	25.116 3894	25.783 3172	27.183 2741	28.676 4857	21
22	26.555 9262	27.298 9835	28.862 8559	30.536 7803	22
23	28.020 6549	28.844 9632	30.584 4273	32.452 8837	23
24	29.511 0164	30.421 8625	32.349 0380	34 426 4702	24
25	31.027 4592	32.030 2997	34.157 7639	36.459 2643	25
26	32.570 4397	33.670 9057	36.011 7080	38.553 0423	26
27	34.140 4224	35.344 3238	37 912 0007	40 709 6335	27
28	35.737 8798	37.051 2103	39.859 8008	42.930 9225	28
29	37.363 2927	38 792 2345	41.856 2958	45 218 8502	29
30	39.017 1503	40.568 0792	43.902 7032	47.575 4157	30
31	40.699 9504	42.379 4408	46.000 2707	50.002 6782	31
32	42.412 1996	44.227 0296	48.150 2775	52.502 7585	32
33	44.154 4130	46.111 5702	50.354 0344	55.077 8413	33
34	45.927 1153	48.033 8016	52.612 8853	57.730 1765	34
35	47.730 8398	49.994 4776	54.928 2074	60.462 0818	35
36	49.566 1295	51.994 3672	57.301 4126	63.275 9443	36
37	51.433 5368	54.034 2545	59.733 9479	66.174 2226	37
38	53.333 6236	56.114 9396	62.227 2966	69.159 4493	38
39	55.266 9621	58.237 2384	64.782 9791	72 234 2328	39
40	57.234 1339	60.401 9832	67.402 5535	75.401 2597	40
41	59.235 7312	62.610 0228	70.087 6174	78 663 2975	41
42	61.272 3565	64.862 2233	72.839 8078	82.023 1965	42
43	63.344 6228	67.159 4678	75.660 8030	85.483 8923	43
44	65.453 1537	69.502 6571	78.552 3231	89.048 4091	44
45	67.598 5839	71.892 7103	81.516 1312	92.719 8614	45
46	69 781 5591	74.330 5645	84.554 0344	96.501 4572	46
47	72.002 7364	76.817 1758	87.667 8853	100.396 5009	47
48	74.262 7843	79.353 5193	90.859 5824	104.408 3960	48
49	76.562 3830	81.940 5897	94.131 0720	108.540 6479	49
50	78.902 2247	84.579 4015	97.484 3488	112.796 8673	50
60	104.675 2159	114.051 5394	135.991 5900	163.053 4368	60
70	135 330 7583	149.977 9111	185.284 1142	230.594 0637	70
80	171.793 8242	193 771 9578	248 382 7126	321 363 0185	80
90	215.164 6172	247.156 6563	329.154 2533	443.348 9037	90
100	266 751 7679	312 232 3059	432 548 6540	607.287 7327	100

Table V—Logarithm of Amount of 1 per Annum

log $s_{\overline{n}|}$

n	1¾%	2%	2½%	3%	n
1	0.00 0000	0.00 0000	0.00 0000	0.00 0000	1
2	0.30 4814	0.30 5351	0.30 6425	0.30 7496	2
3	0.48 4699	0.48 5778	0.48 7933	0.49 0085	3
4	0.61 3443	0.61 5067	0.61 8311	0.62 1553	4
5	0.71 4170	0.71 6341	0.72 0682	0.72 5024	5
6	0.79 7178	0.79 9900	0.80 5347	0.81 0798	6
7	0.86 7963	0.87 1239	0.87 7799	0.88 4368	7
8	0.92 9804	0.93 3638	0.94 1318	0.94 9016	8
9	0.98 4816	0.98 9211	0.99 8020	1.00 6855	9
10	1.03 4444	1.03 9403	1.04 9349	1.05 9332	10
11	1.07 9718	1.08 5245	1.09 6335	1.10 7474	11
12	1.12 1399	1.12 7496	1.13 9739	1.15 2045	12
13	1.16 0064	1.16 6736	1.18 0139	1.19 3620	13
14	1.19 6163	1.20 3412	1.21 7983	1.23 2649	14
15	1.23 0051	1.23 7881	1.25 3627	1.26 9488	15
16	1.26 2016	1.27 0429	1.28 7359	1.30 4423	16
17	1.29 2292	1.30 1292	1.31 9413	1.33 7691	17
18	1.32 1073	1.33 0664	1.34 9983	1.36 9484	18
19	1.34 8522	1.35 8707	1.37 9233	1.39 9966	19
20	1.37 4778	1.38 5559	1.40 7300	1.42 9274	20
21	1.39 9957	1.41 1339	1.43 4302	1.45 7526	21
22	1.42 4161	1.43 6146	1.46 0339	1.48 4823	22
23	1.44 7478	1.46 0070	1.48 5500	1.51 1253	23
24	1.46 9984	1.48 3186	1.50 9861	1.53 6893	24
25	1.49 1746	1.50 5561	1.53 3489	1.56 1808	25
26	1.51 2824	1.52 7255	1.55 6444	1.58 6059	26
27	1.53 3269	1.54 8320	1.57 8777	1.60 9697	27
28	1.55 3129	1.56 8802	1.60 0535	1.63 2770	28
29	1.57 2445	1.58 8745	1.62 1761	1.65 5320	29
30	1.59 1256	1.60 8184	1.64 2491	1.67 7383	30
31	1.60 9594	1.62 7155	1.66 2760	1.69 8993	31
32	1.62 7491	1.64 5688	1.68 2599	1.72 0182	32
33	1.64 4974	1.66 3810	1.70 2034	1.74 0977	33
34	1.66 2069	1.68 1547	1.72 1092	1.76 1403	34
35	1.67 8799	1.69 8922	1.73 9795	1.78 1483	35
36	1.69 5185	1.71 5956	1.75 8165	1.80 1239	36
37	1.71 1246	1.73 2669	1.77 6221	1.82 0689	37
38	1.72 7001	1.74 9078	1.79 3981	1.83 9852	38
39	1.74 2466	1.76 5209	1.81 1461	1.85 8743	39
40	1.75 7655	1.78 1051	1.82 8676	1.87 7379	40
41	1.77 2584	1.79 6644	1.84 5641	1.89 5772	41
42	1.78 7265	1.81 1992	1.86 2369	1.91 3937	42
43	1.80 1710	1.82 7107	1.87 8871	1.93 1884	43
44	1.81 5931	1.84 2001	1.89 5159	1.94 9626	44
45	1.82 9938	1.85 6685	1.91 1244	1.96 7173	45
46	1.84 3741	1.87 1167	1.92 7134	1.98 4534	46
47	1.85 7349	1.88 5458	1.94 2841	2.00 1719	47
48	1.87 0771	1.89 9566	1.95 8371	2.01 8735	48
49	1.88 4015	1.91 3499	1.97 3733	2.03 5592	49
50	1.89 7089	1.92 7265	1.98 8935	2.05 2297	50
60	2.01 9844	2.05 7101	2.13 3512	2.21 2330	60
70	2.13 1397	2.17 6027	2.26 7838	2.36 2848	70
80	2.23 5008	2.28 7291	2.39 5121	2.50 6996	80
90	2.33 2771	2.39 2972	2.51 7400	2.64 6746	90
100	2.42 6107	2.49 4478	2.63 6035	2.78 3394	100

Table V—Amount of 1 per Annum at Compound Interest

$$s_{\overline{n}|} = [(1 + i)^n - 1]/i$$

n	3½%	4%	4½%	5%	n
1	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1
2	2.035 0000	2.040 0000	2.045 0000	2.050 0000	2
3	3.106 2250	3.121 6000	3.137 0250	3.152 5000	3
4	4.214 9429	4.246 4640	4.278 1911	4.310 1250	4
5	5.362 4659	5.416 3226	5.470 7097	5.525 6312	5
6	6 550 1522	6.632 9755	6.716 8917	6.801 9128	6
7	7.779 4075	7.898 2945	8.019 1518	8.142 0085	7
8	9.051 6868	9.214 2263	9.380 0136	9.549 1089	8
9	10.368 4958	10.582 7953	10.802 1142	11.026 5643	9
10	11.731 3932	12.006 1071	12.288 2094	12.577 8925	10
11	13.141 9919	13.486 3514	13.841 1788	14.206 7872	11
12	14.601 9616	15.025 8055	15.464 0318	15.917 1265	12
13	16.113 0303	16.626 8377	17.159 9133	17.712 9828	13
14	17.676 9864	18.291 9112	18.932 1094	19.598 6320	14
15	19.295 6809	20.023 5876	20.784 0543	21.578 5636	15
16	20.971 0297	21.824 5311	22.719 3367	23.657 4918	16
17	22.705 0157	23.697 5124	24.741 7069	25.840 3664	17
18	24.499 6913	25.645 4129	26.855 0837	28.132 3847	18
19	26.357 1805	27.671 2294	29.063 5625	30.539 0039	19
20	28.279 6818	29.778 0786	31.371 4228	33.065 9541	20
21	30.269 4707	31.969 2017	33.783 1368	35.719 2518	21
22	32.328 9022	34.247 9698	36.303 3780	38.505 2144	22
23	34.460 4137	36.617 8886	38.937 0300	41.430 4751	23
24	36.666 5282	39.082 6041	41.689 1963	44.501 9989	24
25	38.949 8567	41.645 9083	44.565 2101	47.727 0988	25
26	41.313 1017	44.311 7446	47.570 6446	51.113 4538	26
27	43.759 0602	47.084 2144	50.711 3236	54.669 1264	27
28	46.290 6273	49.967 5830	53.993 3332	58.402 5828	28
29	48.910 7993	52.966 2863	57.423 0332	62.322 7119	29
30	51.622 6773	56.084 9378	61.007 0697	66.438 8475	30
31	54.429 4710	59.328 3353	64.752 3878	70.760 7899	31
32	57.334 5025	62.701 4687	68.666 2452	75.298 8294	32
33	60.341 2101	66 209 5274	72.756 2263	80.063 7708	33
34	63.453 1524	69.857 9085	77.030 2565	85.066 9594	34
35	66.674 0127	73.652 2249	81.496 6180	90.320 3074	35
36	70.007 6032	77.598 3138	86.163 9658	95.836 3227	36
37	73.457 8693	81.702 2464	91,041 3443	101.628 1389	37
38	77.028 8947	85.970 3363	96.138 2048	107.709 5458	38
39	80.724 9060	90.409 1497	101.464 4240	114.095 0231	39
40	84.550 2777	95.025 5157	107.030 3231	120.799 7742	40
41	88.509 5375	99.826 5363	112.846 6876	127.839 7630	41
42	92.607 3713	104.819 5978	118.924 7885	135.231 7511	42
43	96.848 6293	110.012 3817	125.276 4040	142.993 3387	43
44	101.238 3313	115.412 8770	131.913 8422	151 143 0056	44
45	105.781 6729	121.029 3920	138.849 9651	159.700 1559	45
46	110.484 0314	126.870 5677	146.098 2135	168.685 1637	46
47	115.350 9725	132.945 3904	153.672 6331	178.119 4218	47
48	120.388 2566	139.263 2060	161.587 9016	188.025 3929	48
49	125.601 8456	145.833 7343	169.859 3572	198.426 6626	49
50	130.997 9102	152.667 0837	178.503 0283	209.347 9957	50
60	196.516 8829	237.990 6852	289.497 9540	353.583 7179	60
70	288.937 8646	364.290 4588	461.869 6795	588.528 5107	70
80	419.306 7868	551.244 9767	729.557 6985	971.228 8213	80
90	603.205 0270	827 983 3335	1145 269 0066	1594 607 3010	90
100	862.611 6567	1237.623 7046	1790.855 9563	2610 025 1569	100

Table V—Logarithm of Amount of 1 per Annum

log $s_{\overline{n|}}$

n	3½%	4%	4½%	5%	n
1	0.00 0000	0 00 0000	0 00 0000	0 00 0000	1
2	0.30 8564	0 30 9630	0.31 0693	0 31 1754	2
3	0 49 2233	0 49 4377	0 49 6518	0 49 8655	3
4	0.62 4792	0 62 8027	0 63 1260	0 63 4490	4
5	0.72 9365	0 73 3705	0.73 8044	0 74 2382	5
6	0.81 6251	0 82 1708	0 82 7168	0 83 2631	6
7	0 89 0947	0 89 7533	0 90 4128	0 91 0732	7
8	0.95 6730	0.96 4459	0 97 2203	0 97 9963	8
9	1.01 5716	1 02 4600	1.03 3509	1 04 2440	9
10	1.06 9350	1.07 9402	1.08 9489	1 09 9607	10
11	1 11 8661	1 12 9894	1 14 1173	1 15 2496	11
12	1.16 4411	1 17 6838	1 18 9323	1 20 1865	12
13	1.20 7177	1.22 0810	1 23 4515	1.24 8292	13
14	1.24 7408	1.26 2259	1 27 7199	1 29 2226	14
15	1.28 5460	1.30 1542	1.31 7730	1.33 4023	15
16	1.32 1620	1 33 8945	1 35 6396	1 37 3969	16
17	1.35 6122	1 37 4703	1 39 3430	1 41 2296	17
18	1.38 9161	1.40 9010	1 42 9027	1 44 9207	18
19	1.42 0899	1 44 2029	1.46 3349	1 48 4855	19
20	1.45 1475	1 47 3897	1 49 6534	1 51 9381	20
21	1.48 1005	1 50 4732	1 52 8700	1 55 2902	21
22	1.50 9510	1 53 4635	1 55 9947	1 58 5520	22
23	1.53 7320	1 56 3693	1 59 0363	1 61 7320	23
24	1.56 4270	1 59 1983	1.62 0024	1 64 8380	24
25	1.59 0506	1 61 9572	1 64 8996	1 67 8765	25
26	1.61 6088	1 64 6519	1 67 7339	1 70 8535	26
27	1.64 1068	1 67 2875	1 70 5105	1 73 7742	27
28	1.66 5493	1 69 8688	1.73 2340	1.76 6432	28
29	1 68 9405	1.72 4000	1 75 9086	1 79 4646	29
30	1.71 2841	1 74 8846	1 78 5380	1.82 2422	30
31	1.73 5834	1 77 3262	1 81 1256	1 84 9793	31
32	1.75 8416	1 79 7278	1 83 6743	1 87 6788	32
33	1.78 0614	1.82 0920	1 86 1870	1 90 3436	33
34	1.80 2453	1 84 4216	1 88 6661	1 92 9761	34
35	1.82 3957	1.86 7186	1 91 1140	1.95 5785	35
36	1.84 5145	1 88 9852	1 93 5326	1 98 1530	36
37	1 86 6038	1.91 2234	1 95 9239	2 00 7014	37
38	1.88 6654	1 93 4349	1 98 2896	2 03 2254	38
39	1.90 7008	1.95 6212	2 00 6314	2 05 7267	39
40	1.92 7115	1.97 7840	2 02 9507	2 08 2066	40
41	1.94 6990	1 99 9246	2 05 2489	2.10 6666	41
42	1.96 6646	2.02 0442	2 07 5272	2 13 1079	42
43	1.98 6093	2 04 1442	2 09 7869	2 15 5316	43
44	2.00 5345	2 06 2254	2 12 0290	2 17 9388	44
45	2.02 4410	2 08 2891	2.14 2546	2 20 3305	45
46	2.04 3299	2 10 3361	2 16 4645	2 22 7077	46
47	2 06 2021	2 12 3673	2 18 6597	2 25 0711	47
48	2.08 0584	2 14 3836	2 20 8409	2.27 4217	48
49	2.09 8996	2 16 3858	2 23 0089	2 29 7600	49
50	2.11 7264	2.18 3745	2 25 1646	2.32 0869	50
60	2.29 3400	2 37 6560	2 46 1646	2 54 8492	60
70	2.46 0804	2 56 1448	2 66 4520	2 76 9767	70
80	2.62 2532	2 74 1345	2 86 3060	2 98 7322	80
90	2.78 0465	2 91 8021	3 05 8908	3 20 2654	90
100	2.93 5815	3.09 2589	3 25 3061	3.41 6645	100

Table V—Amount of 1 per Annum at Compound Interest

$$s_{\overline{n}|} = [(1 + i)^n - 1]/i$$

n	5½%	6%	7%	8%	*n*
1	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1
2	2.055 0000	2.060 0000	2.070 0000	2.080 0000	2
3	3.168 0250	3.183 6000	3.214 9000	3.246 4000	3
4	4.342 2664	4.374 6160	4.439 9430	4.506 1120	4
5	5.581 0910	5.637 0930	5.750 7390	5.866 6010	5
6	6.888 0510	6.975 3185	7.153 2907	7.335 9290	6
7	8.266 8938	8.393 8376	8.654 0211	8.922 8034	7
8	9.721 5730	9.897 4679	10.259 8026	10.636 6276	8
9	11.256 2595	11.491 3160	11.977 9887	12.487 5578	9
10	12.875 3538	13.180 7949	13.816 4480	14.486 5625	10
11	14.583 4982	14.971 6426	15.783 5993	16.645 4875	11
12	16.385 5907	16.869 9412	17.888 4513	18.977 1265	12
13	18.286 7981	18.882 1377	20.140 6429	21.495 2966	13
14	20.292 5720	21.015 0659	22.550 4879	24.214 9203	14
15	22.408 6635	23.275 9699	25.129 0220	27.152 1139	15
16	24.641 1400	25.672 5281	27.888 0536	30.324 2830	16
17	26.996 4027	28.212 8798	30.840 2173	33.750 2257	17
18	29.481 2048	30.905 6525	33.999 0325	37.450 2437	18
19	32.102 6711	33.759 9917	37.378 9648	41.446 2632	19
20	34.868 3180	36.785 5912	40.995 4923	45.761 9643	20
21	37.786 0755	39.992 7267	44.865 1768	50.422 9214	21
22	40.864 3097	43.392 2903	49.005 7392	55.456 7552	22
23	44.111 8467	46.995 8277	53.436 1409	60.893 2956	23
24	47.537 9983	50.815 5774	58.176 6708	66.764 7592	24
25	51.152 5882	54.864 5120	63.249 0377	73.105 9400	25
26	54.965 9805	59.156 3827	68.676 4704	79.954 4151	26
27	58.989 1094	63.705 7657	74.483 8233	87.350 7684	27
28	63.233 5105	68.528 1116	80.697 6909	95.338 8298	28
29	67.711 3535	73.639 7983	87.346 5293	103.965 9362	29
30	72.435 4780	79.058 1862	94.460 7863	113.283 2111	30
31	77.419 4293	84.801 6774	102.073 0414	123.345 8680	31
32	82.677 4979	90.889 7780	110.218 1543	134.213 5374	32
33	88.224 7603	97.343 1647	118.933 4251	145.950 6204	33
34	94.077 1221	104.183 7546	128.258 7648	158.626 6701	34
35	100.251 3638	111.434 7799	138.236 8784	172.316 8037	35
36	106.765 1888	119.120 8667	148.913 4598	187.102 1480	36
37	113.637 2742	127.268 1187	160.337 4020	203.070 3198	37
38	120.887 3242	135.904 2058	172.561 0202	220.315 9454	38
39	128.536 1271	145.058 4581	185.640 2916	238.941 2210	39
40	136.605 6141	154.761 9656	199.635 1120	259.056 5187	40
41	145.118 9228	165.047 6836	214.609 5698	280.781 0402	41
42	154.100 4636	175.950 5446	230.632 2397	304.243 5234	42
43	163.575 9891	187.507 5772	247.776 4965	329.583 0053	43
44	173.572 6685	199.758 0319	266.120 8513	356.949 6457	44
45	184.119 1653	212.743 5138	285.749 3108	386.505 6174	45
46	195.245 7194	226.508 1246	306.751 7626	418.426 0668	46
47	206.984 2339	241.098 6121	329.224 3860	452.900 1521	47
48	219.368 3668	256.564 5288	353.270 0930	490.132 1643	48
49	232.433 6270	272.958 4006	378.998 9995	530.342 7374	49
50	246.217 4764	290.335 9046	406.528 9295	573.770 1564	50
60	433.450 3717	533.128 1809	813.520 3834	1253.213 2958	60
70	753.271 2042	967.932 1696	1614.134 1743	2720.080 0738	70
80	1299.571 3869	1746.599 8914	3189.062 6797	5886.935 4283	80
90	2232.731 0166	3141.075 1872	6287.185 4268	12723.938 6160	90
100	3826.702 4668	5638.368 0586	12381.661 7938	27484.515 7043	100

Table V—Logarithm of Amount of 1 per Annum

log $s_{\overline{n}|}$

n	5½%	6%	7%	8%	*n*
1	0.00 0000	0.00 0000	0.00 0000	0.00 0000	1
2	0.31 2812	0.31 3867	0.31 5970	0.31 8063	2
3	0.50 0789	0.50 2918	0.50 7167	0.51 1402	3
4	0.63 7716	0.64 0940	0.64 7377	0.65 3802	4
5	0.74 6719	0.75 1055	0.75 9724	0.76 8387	5
6	0.83 8096	0.84 3564	0.85 4506	0.86 5455	6
7	0.91 7342	0.92 3961	0.93 7218	0.95 0301	7
8	0.98 7737	0.99 5524	1.01 1139	1.02 6804	8
9	1.05 1394	1.06 0370	1.07 8384	1.09 6478	9
10	1.10 9759	1.11 9942	1.14 0396	1.16 0965	10
11	1.16 3862	1.17 5269	1.19 8206	1.22 1297	11
12	1.21 4462	1.22 7114	1.25 2573	1.27 8230	12
13	1.26 2138	1.27 6051	1.30 4073	1.33 2343	13
14	1.30 7337	1.32 2531	1.35 3156	1.38 4083	14
15	1.35 0416	1.36 6908	1.40 0176	1.43 3804	15
16	1.39 1661	1.40 9469	1.44 5418	1.48 1791	16
17	1.43 1306	1.45 0147	1.48 9117	1.52 8277	17
18	1.46 9545	1.49 0038	1.53 1467	1.57 3455	18
19	1.50 6541	1.52 8402	1.57 2627	1.61 7485	19
20	1.54 2431	1.56 5677	1.61 2736	1.66 0505	20
21	1.57 7332	1.60 1981	1.65 1909	1.70 2628	21
22	1.61 1344	1.63 7413	1.69 0247	1.74 3954	22
23	1.64 4555	1.67 2059	1.72 7825	1.78 4569	23
24	1.67 7041	1.70 5997	1.76 4749	1.82 4547	24
25	1.70 8867	1.73 9292	1.80 1054	1.86 3953	25
26	1.74 0940	1.77 2002	1.83 6808	1.90 2842	26
27	1.77 0772	1.80 4179	1.87 2062	1.94 1267	27
28	1.80 0947	1.83 5869	1.90 6861	1.97 9270	28
29	1.83 0661	1.86 7113	1.94 1246	2.01 6891	29
30	1.85 9951	1.89 7947	1.97 5252	2.05 4166	30
31	1.88 8850	1.92 8404	2.00 8911	2.09 1125	31
32	1.91 7387	1.95 8515	2.04 2253	2.12 7796	32
33	1.94 5590	1.98 8305	2.07 5304	2.16 4206	33
34	1.97 3484	2.01 7800	2.10 8088	2.20 0376	34
35	2.00 1090	2.04 7021	2.14 0624	2.23 6328	35
36	2.02 8430	2.07 5988	2.17 2934	2.27 2079	36
37	2.05 5521	2.10 4720	2.20 5035	2.30 7646	37
38	2.08 2381	2.13 3233	2.23 6943	2.34 3046	38
39	2.10 9025	2.16 1543	2.26 8672	2.37 8291	39
40	2.13 5469	2.18 9664	2.30 0237	2.41 3395	40
41	2.16 1724	2.21 7609	2.33 1649	2.44 8368	41
42	2.18 7804	2.24 5391	2.36 2920	2.48 3221	42
43	2.21 3720	2.27 3019	2.39 4060	2.51 7965	43
44	2.23 9481	2.30 0504	2.42 5079	2.55 2607	44
45	2.26 5099	2.32 7856	2.45 5985	2.58 7156	45
46	2.29 0582	2.35 5084	2.48 6787	2.62 1619	46
47	2.31 5937	2.38 2195	2.51 7492	2.65 6002	47
48	2.34 1174	2.40 9197	2.54 8107	2.69 0313	48
49	2.36 6299	2.43 6096	2.57 8628	2.72 4557	49
50	2.39 1319	2.46 2901	2.60 9091	2.75 8738	50
60	2.63 6939	2.72 6832	2.91 0368	3.09 8025	60
70	2.87 6951	2.98 5845	3.20 7940	3.43 4582	70
80	3.11 3800	3.24 2193	3.50 3663	3.76 9889	80
90	3.34 8836	3.49 7078	3.79 8456	4.10 4622	90
100	3.58 2825	3.75 1153	4.09 2779	4.43 9068	100

Table VI—Present Value of 1 per Annum

$$a_{\overline{n}|} = (1 - v^n)/i$$

n	½%	1%	1¼%	1½%	n
1	0.995 0249	0.990 0990	0.987 6543	0.985 2217	1
2	1.985 0994	1.970 3951	1.963 1154	1.955 8834	2
3	2.970 2481	2.940 9852	2.926 5337	2.912 2004	3
4	3.950 4957	3.901 9656	3.878 0580	3.854 3846	4
5	4.925 8663	4.853 4312	4.817 8350	4.782 6450	5
6	5.896 3844	5.795 4765	5.746 0099	5.697 1872	6
7	6.862 0740	6.728 1945	6.662 7258	6.598 2140	7
8	7.822 9592	7.651 6778	7.568 1243	7.485 9251	8
9	8.779 0639	8.566 0176	8.462 3450	8.360 5173	9
10	9.730 4119	9.471 3045	9.345 5259	9.222 1846	10
11	10.677 0267	10.367 6282	10.217 8034	10.071 1178	11
12	11.618 9321	11.255 0775	11.079 3120	10.907 5052	12
13	12.556 1513	12.133 7401	11.930 1847	11.731 5322	13
14	13.488 7078	13.003 7030	12.770 5527	12.543 3815	14
15	14.416 6246	13.865 0525	13.600 5459	13.343 2330	15
16	15.339 9250	14.717 8738	14.420 2923	14.131 2640	16
17	16.258 6319	15.562 2513	15.229 9183	14.907 6493	17
18	17.172 7680	16.398 2686	16.029 5489	15.672 5609	18
19	18.082 3562	17.226 0085	16.819 3076	16.426 1684	19
20	18.987 4191	18.045 5530	17.599 3161	17.168 6388	20
21	19.887 9793	18.856 9831	18.369 6949	17.900 1367	21
22	20.784 0590	19.660 3793	19.130 5629	18.620 8244	22
23	21.675 6806	20.455 8211	19.882 0374	19.330 8614	23
24	22.562 8662	21.243 3873	20.624 2345	20.030 4054	24
25	23.445 6380	22.023 1557	21.357 2687	20.719 6112	25
26	24.324 0179	22.795 2037	22.081 2530	21.398 6317	26
27	25.198 0278	23.559 6076	22.796 2993	22.067 6175	27
28	26.067 6894	24.316 4432	23.502 5178	22.726 7167	28
29	26.933 0242	25.065 7853	24.200 0176	23.376 0756	29
30	27.794 0540	25.807 7082	24.888 9062	24.015 8380	30
31	28.650 8000	26.542 2854	25.569 2901	24.646 1458	31
32	29.503 2835	27.269 5895	26.241 2742	25.267 1387	32
33	30.351 5259	27.989 6925	26.904 9622	25.878 9544	33
34	31.195 5482	28.702 6659	27.560 4564	26.481 7285	34
35	32.035 3713	29.408 5801	28.207 8582	27.075 5946	35
36	32.871 0162	30.107 5050	28.847 2674	27.660 6843	36
37	33.702 5037	30.799 5099	29.478 7826	28.237 1274	37
38	34.529 8544	31.484 6633	30.102 5013	28.805 0516	38
39	35.353 0890	32.163 0330	30.718 5198	29.364 5829	39
40	36.172 2279	32.834 6861	31.326 9332	29.915 8452	40
41	36.987 2914	33.499 6892	31.927 8352	30.458 9608	41
42	37.798 2999	34.158 1081	32.521 3187	30.994 0500	42
43	38.605 2735	34.810 0081	33.107 4753	31.521 2316	43
44	39.408 2324	35.455 4535	33.686 3954	32.040 6222	44
45	40.207 1964	36.094 5084	34.258 1683	32.552 3372	45
46	41.002 1855	36.727 2361	34.822 8822	33.056 4898	46
47	41.793 2194	37.353 6991	35.380 6244	33.553 1920	47
48	42.580 3178	37.973 9595	35.931 4809	34.042 5536	48
49	43.363 5003	38.588 0787	36.475 5367	34.524 6834	49
50	44.142 7863	39.196 1175	37.012 8757	34.999 6881	50
60	51.725 5608	44.955 0384	42.034 5918	39.380 2689	60
70	58.939 4176	50.168 5143	46.469 6756	43.154 8718	70
80	65.802 3054	54.888 2061	50.386 6571	46.407 3235	80
90	72.331 2996	59.160 8815	53.846 0604	49.209 8545	90
100	78.542 6448	63.028 8788	56.901 3394	51.624 7037	100

Table VI—Logarithm of Present Value of 1 per Annum

log $a_{\overline{n}|}$

n	½%	1%	1¼%	1½%	n
1	9 99 7834	9 99 5679	9 99 4605	9 99 3534	1
2	0.29 7782	0 29 4553	0.29 2946	0 29 1343	2
3	0 47 2793	0.46 8493	0 46 6354	0 46 4221	3
4	0.59 6652	0 59 1283	0 58 8614	0.58 5955	4
5	0.69 2483	0.68 6049	0.68 2852	0.67 9668	5
6	0 77 0586	0.76 3089	0 75 9366	0 75 5660	6
7	0.83 6455	0.82 7899	0.82 3652	0 81 9426	7
8	0.89 3371	0.88 3757	0 87 8988	0.87 4245	8
9	0 94 3448	0.93 2779	0.92 7491	0 92 2233	9
10	0.98 8131	0.97 6410	0.97 0604	0.96 4834	10
11	1.02 8450	1 01 5679	1.00 9357	1.00 3078	11
12	1.06 5166	1.05 1348	1.04 4513	1.03 7725	12
13	1.09 8857	1.08 3995	1.07 6647	1 06 9355	13
14	1 12 9970	1 11 4067	1 10 6210	1.09 8415	14
15	1 15 8864	1 14 1922	1 13 3556	1 12 5261	15
16	1 18 5823	1 16 7845	1 15 8974	1 15 0181	16
17	1.21 1084	1 19 2072	1 18 2698	1 17 3409	17
18	1 23 4840	1 21 4798	1.20 4921	1 19 5140	18
19	1.25 7255	1.23 6185	1.22 5808	1.21 5536	19
20	1.27 8466	1.25 6370	1.24 5496	1.23 4736	20
21	1.29 8591	1.27 5472	1.26 4102	1 25 2856	21
22	1.31 7730	1.29 3592	1.28 1728	1.26 9999	22
23	1.33 5973	1.31 0817	1.29 8461	1.28 6251	23
24	1.35 3394	1.32 7224	1.31 4378	1.30 1690	24
25	1.37 0062	1.34 2880	1.32 9546	1.31 6382	25
26	1.38 6035	1 35 7843	1.34 4024	1 33 0386	26
27	1.40 1367	1 37 2168	1.35 7864	1 34 3755	27
28	1.41 6103	1.38 5900	1.37 1114	1.35 6537	28
29	1.43 0285	1 39 9081	1.38 3816	1.36 8772	29
30	1.44 3952	1.41 1749	1.39 6006	1.38 0498	30
31	1.45 7137	1 42 3938	1.40 7719	1.39 1749	31
32	1.46 9870	1.43 5679	1.41 8985	1.40 2556	32
33	1.48 2181	1 44 6998	1.42 9832	1.41 2947	33
34	1.49 4093	1.45 7922	1.44 0286	1 42 2946	34
35	1.50 5630	1.46 8474	1.45 0370	1.43 2578	35
36	1.51 6813	1 47 8675	1 46 0105	1.44 1863	36
37	1.52 7662	1.48 8544	1.46 9509	1.45 0821	37
38	1.53 8195	1 49 8099	1.47 8603	1.45 9469	38
39	1.54 8427	1.50 7357	1.48 7400	1.46 7824	39
40	1.55 8375	1.51 6333	1.49 5918	1.47 5901	40
41	1.56 8053	1.52 5041	1.50 4169	1.48 3715	41
42	1.57 7472	1.53 3494	1.51 2168	1.49 1278	42
43	1.58 6647	1.54 1704	1.51 9926	1.49 8603	43
44	1.59 5587	1.54 9683	1.52 7455	1 50 5701	44
45	1.60 4304	1.55 7441	1.53 4764	1.51 2582	45
46	1.61 2807	1.56 4988	1.54 1865	1.51 9257	46
47	1.62 1106	1.57 2334	1.54 8765	1.52 5734	47
48	1.62 9209	1.57 9486	1.55 5475	1 53 2022	48
49	1.63 7124	1.58 6453	1.56 2002	1.53 8130	49
50	1.64 4860	1.59 3243	1.56 8353	1.54 4064	50
60	1.71 3705	1.65 2778	1.62 3607	1.59 5279	60
70	1.77 0406	1.70 0431	1.66 7170	1.63 5030	70
80	1.81 8241	1.73 9479	1 70 2316	1.66 6587	80
90	1.85 9326	1 77 2035	1.73 1154	1.69 2052	90
100	1.89 5106	1.79 9540	1 75 5124	1 71 2858	100

Table VI—Present Value of 1 per Annum

$$a_{\overline{n}|} = (1 - v^n)/i$$

n	1¾%	2%	2½%	3%	n
1	0.982 8010	0.980 3922	0.975 6098	0.970 8738	1
2	1.948 6988	1.941 5609	1.927 4242	1.913 4697	2
3	2.897 9840	2.883 8833	2.856 0236	2.828 6114	3
4	3.830 9425	3.807 7287	3.761 9742	3.717 0984	4
5	4.747 8551	4.713 4595	4.645 8285	4.579 7072	5
6	5.648 9976	5.601 4309	5.508 1254	5.417 1914	6
7	6.534 6414	6.471 9911	6.349 3906	6.230 2830	7
8	7.405 0530	7.325 4814	7.170 1372	7.019 6922	8
9	8.260 4943	8.162 2367	7.970 8655	7.786 1089	9
10	9.101 2229	8.982 5850	8.752 0639	8.530 2028	10
11	9.927 4918	9.786 8480	9.514 2087	9.252 6241	11
12	10.739 5497	10.575 3412	10.257 7646	9.954 0040	12
13	11.537 6410	11.348 3737	10.983 1850	10.634 9553	13
14	12.322 0059	12.106 2488	11.690 9122	11.296 0731	14
15	13.092 8805	12.849 2635	12.381 3777	11.937 9351	15
16	13.850 4968	13.577 7093	13.055 0027	12.561 1020	16
17	14.595 0828	14.291 8719	13.712 1977	13.166 1185	17
18	15.326 8627	14.992 0313	14.353 3636	13.753 5131	18
19	16 046 0567	15.678 4620	14.978 8913	14 323 7991	19
20	16.752 8813	16.351 4333	15.589 1623	14.877 4749	20
21	17.447 5492	17.011 2092	16.184 5486	15.415 0241	21
22	18.130 2695	17.658 0482	16.765 4132	15.936 9166	22
23	18.801 2476	18.292 2041	17.332 1105	16.443 6084	23
24	19.460 6856	18.913 9256	17.884 9858	16.935 5421	24
25	20.108 7820	19.523 4565	18.424 3764	17.413 1477	25
26	20.745 7317	20.121 0358	18.950 6111	17.876 8424	26
27	21.371 7264	20.706 8978	19.464 0109	18.327 0315	27
28	21.986 9547	21.281 2724	19.964 8887	18.764 1082	28
29	22.591 6017	21.844 3847	20.453 5499	19.188 4546	29
30	23.185 8493	22.396 4556	20.930 2926	19.600 4414	30
31	23.769 8765	22.937 7015	21.395 4074	20.000 4285	31
32	24.343 8590	23.468 3348	21.849 1780	20.388 7655	32
33	24.907 9695	23.988 5636	22.291 8809	20.765 7918	33
34	25.462 3779	24.498 5917	22.723 7863	21.131 8367	34
35	26.007 2510	24.998 6193	23.145 1573	21.487 2201	35
36	26.542 7528	25.488 8425	23.556 2511	21.832 2525	36
37	27.069 0445	25.969 4534	23.957 3181	22.167 2354	37
38	27.586 2846	26.440 6406	24.348 6030	22.492 4616	38
39	28.094 6286	26.902 5888	24.730 3444	22.808 2151	39
40	28.594 2295	27.355 4792	25.102 7751	23.114 7720	40
41	29.085 2379	27.799 4895	25.466 1220	23.412 4000	41
42	29.567 8014	28.234 7936	25.820 6068	23.701 3592	42
43	30.042 0652	28.661 5623	26.166 4457	23.981 9021	43
44	30.508 1722	29.079 9631	26.503 8495	24.254 2739	44
45	30.966 2626	29.490 1599	26.833 0239	24.518 7125	45
46	31.416 4743	29.892 3136	27.154 1696	24.775 4491	46
47	31.858 9428	30.286 5820	27.467 4826	25.024 7078	47
48	32.293 8013	30.673 1196	27.773 1537	25.266 7066	48
49	32.721 1806	31.052 0780	28.071 3695	25.501 6569	49
50	33.141 2095	31.423 6059	28.362 3117	25.729 7640	50
60	36.963 9855	34.760 8867	30.908 6565	27.675 5637	60
70	40.177 9027	37.498 6193	32.897 8570	29.123 4214	70
80	42.879 9347	39.744 5136	34.451 8172	30.200 7634	80
90	45.151 6104	41.586 9292	35.665 7685	31.002 4071	90
100	47.061 4730	43.098 3516	36.614 1053	31.598 9053	100

Table VI—Logarithm of Present Value of 1 per Annum

$\log a_{\overline{n}|}$

n	1¾%	2%	2½%	3%	n
1	9.99 2466	9.99 1400	9.98 9276	9.98 7163	1
2	0.28 9745	0.28 8151	0.28 4977	0.28 1822	2
3	0.46 2096	0.45 9978	0.45 5762	0.45 1573	3
4	0.58 3306	0.58 0666	0 57 5416	0.57 0204	4
5	0.67 6497	0.67 3340	0 66 7063	0.66 0838	5
6	0.75 1971	0.74 8299	0.74 1004	0.73 3774	6
7	0.81 5222	0.81 1038	0 80 2732	0.79 4508	7
8	0.86 9528	0.86 4836	0.85 5527	0.84 6318	8
9	0.91 7006	0.91 1809	0.90 1505	0 89 1320	9
10	0.95 9100	0.95 3401	0.94 2110	0.93 0959	10
11	0.99 6840	0.99 0643	0.97 8373	0.96 6265	11
12	1.03 0986	1.02 4294	1.01 1053	0 99 7998	12
13	1.06 2117	1.05 4934	1.04 0728	1.02 6736	13
14	1.09 0681	1.08 3010	1.06 7848	1.05 2927	14
15	1.11 7035	1.10 8878	1.09 2769	1.07 6929	15
16	1.14 1465	1.13 2827	1 11 5770	1.09 9028	16
17	1.16 4207	1.15 5089	1.13 7107	1.11 9458	17
18	1.18 5453	1.17 5860	1.15 6954	1.13 8414	18
19	1.20 5368	1.19 5303	1.17 5480	1.15 6058	19
20	1.22 4090	1.21 3556	1.19 2823	1.17 2529	20
21	1.24 1734	1.23 0735	1.20 9101	1.18 7944	21
22	1.25 8404	1.24 6943	1 22 4414	1.20 2404	22
23	1.27 4187	1.26 2266	1.23 8851	1.21 5997	23
24	1.28 9158	1.27 6782	1.25 2489	1.22 8799	24
25	1.30 0386	1.29 0557	1.26 5393	1.24 0877	25
26	1.31 6929	1.30 3650	1 27 7623	1.25 2291	26
27	1.32 9840	1.31 6115	1.28 9232	1.26 3092	27
28	1.34 2165	1.32 7998	1.30 0267	1.27 3328	28
29	1.35 3947	1.33 9340	1.31 0769	1 28 3040	29
30	1.36 5223	1.35 0179	1.32 0775	1.29 2266	30
31	1.37 6027	1.36 0550	1 33 0321	1.30 1039	31
32	1.38 6389	1.37 0482	1 33 9435	1.30 9391	32
33	1.39 6338	1.08 0004	1 34 8147	1.31 7348	33
34	1.40 5899	1.38 9141	1.35 6481	1.32 4937	34
35	1.41 5094	1.39 7916	1 36 4460	1.33 2180	35
36	1.42 3946	1.40 6350	1.37 2106	1.33 9099	36
37	1.43 2473	1.41 4463	1.37 9438	1.34 5712	37
38	1.44 0693	1.42 2272	1.38 6474	1.35 2037	38
39	1.44 8623	1.42 9794	1.39 3230	1.35 8091	39
40	1.45 6278	1.43 7044	1.39 9722	1.36 3890	40
41	1.46 3673	1.44 4037	1.40 5963	1.36 9446	41
42	1.47 0819	1.45 0785	1.41 1966	1.37 4773	42
43	1.47 7730	1.45 7300	1.41 7745	1.37 9884	43
44	1.48 4416	1.46 3594	1.42 3309	1.38 4788	44
45	1.49 0889	1.46 9677	1.42 8670	1.38 9498	45
46	1.49 7157	1.47 5560	1.43 3837	1.39 4022	46
47	1.50 3231	1.48 1250	1.43 8819	1 39 8369	47
48	1.50 9119	1.48 6758	1.44 3625	1 40 2549	48
49	1.51 4829	1.49 2091	1.44 8264	1 40 6568	49
50	1.52 0368	1.49 7256	1.45 2742	1.41 0436	50
60	1.56 7779	1.54 1091	1.49 0080	1.44 2096	60
70	1.60 3987	1.57 4015	1 51 7168	1.46 4242	70
80	1.63 2254	1.59 9277	1 53 7212	1.48 0018	80
90	1.65 4673	1.61 8957	1.55 2252	1.49 1395	90
100	1.67 2666	1.63 4461	1 56 3648	1 49 9672	100

Table VI—Present Value of 1 per Annum

$$a_{\overline{n|}} = (1 - v^n)/i$$

n	3½%	4%	4½%	5%	n
1	0.966 1836	0.961 5385	0 956 9378	0.952 3810	1
2	1.899 6943	1.886 0947	1.872 6678	1.859 4104	2
3	2.801 6370	2.775 0910	2.748 9644	2.723 2480	3
4	3.673 0792	3 629 8952	3.587 5257	3 545 9505	4
5	4.515 0524	4.451 8223	4.389 9767	4.329 4767	5
6	5 328 5530	5.242 1369	5.157 8725	5.075 6921	6
7	6 114 5440	6.002 0547	5.892 7009	5.786 3734	7
8	6.873 9555	6.732 7449	6.595 8861	6.463 2128	8
9	7.607 6865	7.435 3316	7.268 7905	7.107 8217	9
10	8.316 6053	8.110 8958	7.912 7182	7.721 7349	10
11	9.001 5510	8 760 4767	8.528 9169	8.306 4142	11
12	9.663 3343	9 385 0738	9.118 5808	8.863 2516	12
13	10.302 7385	9.985 6478	9.682 8524	9.393 5730	13
14	10.920 5203	10.563 1229	10.222 8253	9.898 6409	14
15	11.517 4109	11.118 3874	10.739 5457	10.379 6580	15
16	12.094 1168	11.652 2956	11.234 0150	10.837 7696	16
17	12.651 3206	12.165 6689	11.707 1914	11.274 0662	17
18	13.189 6817	12.659 2970	12.159 9918	11.689 5869	18
19	13.709 8374	13.133 9394	12.593 2936	12.085 3209	19
20	14.212 4033	13.590 3263	13.007 9365	12.462 2103	20
21	14.697 9742	14.029 1599	13.404 7239	12.821 1527	21
22	15.167 1248	14.451 1153	13.784 4248	13.163 0026	22
23	15.620 4105	14.856 8417	14.147 7749	13.488 5739	23
24	16.058 3676	15.246 9631	14.495 4784	13.798 6418	24
25	16.481 5146	15.622 0799	14.828 2090	14.093 9446	25
26	16.890 3523	15.982 7692	15.146 6114	14.375 1853	26
27	17.285 3645	16.329 5857	15.451 3028	14.643 0336	27
28	17.667 0188	16.663 0632	15.742 8735	14.898 1273	28
29	18 035 7670	16.983 7146	16.021 8885	15.141 0736	29
30	18.392 0454	17.292 0333	16.288 8885	15.372 4510	30
31	18.736 2758	17.588 4936	16.544 3910	15.592 8105	31
32	19.068 8655	17.873 5515	16.788 8909	15.802 6767	32
33	19.390 2082	18.147 6457	17.022 8621	16.002 5492	33
34	19.700 6842	18.411 1978	17.246 7580	16.192 9040	34
35	20.000 6611	18.664 6132	17.461 0124	16.374 1943	35
36	20.290 4938	18.908 2820	17.666 0406	16.546 8517	36
37	20.570 5254	19.142 5788	17.862 2398	16.711 2873	37
38	20.841 0874	19.367 8642	18.049 9902	16.867 8927	38
39	21.102 4999	19.584 4848	18.229 6557	17.017 0407	39
40	21.355 0723	19.792 7739	18.401 5844	17.159 0864	40
41	21.599 1037	19.993 0518	18.566 1095	17.294 3680	41
42	21.834 8828	20.185 6267	18.723 5498	17.423 2076	42
43	22.062 6887	20.370 7949	18 874 2103	17 545 9120	43
44	22.282 7910	20.548 8413	19.018 3831	17.662 7733	44
45	22.495 4503	20.720 0397	19.156 3474	17.774 0698	45
46	22.700 9181	20.884 6536	19.288 3707	17 880 0665	46
47	22.899 4378	21.042 9361	19.414 7088	17.981 0157	47
48	23.091 2443	21.195 1309	19.535 6065	18.077 1578	48
49	23.276 5645	21.341 4720	19.651 2981	18.168 7217	49
50	23.455 6179	21.482 1846	19.762 0078	18.255 9255	50
60	24.944 7341	22.623 4900	20.638 0220	18.929 2895	60
70	26.000 3966	23.394 5150	21.202 1119	19.342 6766	70
80	26.748 7757	23.915 3918	21.565 3449	19.596 4605	80
90	27.279 3156	24.267 2776	21.799 2407	19.752 2617	90
100	27.655 4254	24.504 9990	21 949 8527	19 847 9102	100

Table VI—Logarithm of Present Value of 1 per Annum

log $a_{\overline{n}|}$

n	3½%	4%	4½%	5%	n
1	9.98 5060	9.98 2967	9.98 0884	9.97 8811	1
2	0.27 8684	0.27 5563	0.27 2461	0.26 9375	2
3	0.44 7412	0.44 3277	0.43 9169	0.43 5087	3
4	0.56 5030	0.55 9894	0.55 4795	0.54 9733	4
5	0.65 4663	0.64 8538	0.64 2462	0.63 6435	5
6	0.72 6609	0.71 9508	0.71 2471	0.70 5495	6
7	0.78 6364	0.77 8300	0.77 0314	0.76 2406	7
8	0.83 7207	0.82 8192	0.81 9273	0.81 0448	8
9	0.88 1253	0.87 1300	0.86 1462	0.85 1737	9
10	0.91 9946	0.90 9069	0.89 8326	0.88 7715	10
11	0.95 4317	0.94 2528	0.93 0894	0.91 9414	11
12	0.98 5127	0.97 2438	0.95 9927	0.94 7593	12
13	1.01 2953	0.99 9376	0.98 6003	0.97 2831	13
14	1.03 8243	1.02 3792	1.00 9571	0.99 5576	14
15	1.06 1355	1.04 6042	1.03 0986	1.01 6183	15
16	1.08 2574	1.06 6411	1.05 0535	1.03 4940	16
17	1.10 2136	1.08 5136	1.06 8453	1.05 2081	17
18	1.12 0234	1.10 2410	1.08 4933	1.06 7799	18
19	1.13 7032	1.11 8395	1.10 0139	1.08 2258	19
20	1.15 2668	1.13 3230	1.11 4208	1.09 5595	20
21	1.16 7257	1.14 7032	1.12 7258	1.10 7927	21
22	1.18 0903	1.15 9901	1.13 9389	1.11 9355	22
23	1.19 3692	1.17 1926	1.15 0688	1.12 9966	23
24	1.20 5701	1.18 3183	1.16 1233	1.13 9836	24
25	1.21 6997	1.19 3739	1.17 1089	1.14 9033	25
26	1.22 7639	1.20 3652	1.18 0315	1.15 7613	26
27	1.23 7679	1.21 2975	1.18 8965	1.16 5631	27
28	1.24 7163	1.22 1755	1.19 7084	1.17 3132	28
29	1.25 6135	1.23 0033	1.20 4714	1.18 0157	29
30	1.26 4630	1.23 7846	1.21 1891	1.18 6743	30
31	1.27 2683	1.24 5229	1.21 8651	1.19 2924	31
32	1.28 0325	1.25 2211	1.22 5022	1.19 8731	32
33	1.28 7582	1.25 8820	1.23 1033	1.20 4189	33
34	1.29 4481	1.26 5082	1.23 6707	1.20 9325	34
35	1.30 1044	1.27 1019	1.24 2069	1.21 4160	35
36	1.30 7293	1.27 6652	1.24 7139	1.21 8715	36
37	1.31 3254	1.28 2000	1.25 1936	1.22 3010	37
38	1.31 8920	1.28 7082	1.25 6477	1.22 7061	38
39	1.32 4334	1.29 1912	1.26 0778	1.23 0884	39
40	1.32 9501	1.29 6507	1.26 4855	1.23 4494	40
41	1.33 4436	1.30 0879	1.26 8721	1.23 7905	41
42	1.33 9151	1.30 5042	1.27 2388	1.24 1128	42
43	1.34 3658	1.30 9008	1.27 5869	1.24 4176	43
44	1.34 7970	1.31 2787	1.27 9174	1.24 7059	44
45	1.35 2095	1.31 6391	1.28 2313	1.24 9787	45
46	1.35 6043	1.31 9827	1.28 5296	1.25 2369	46
47	1.35 9825	1.32 3106	1.28 8131	1.25 4814	47
48	1.36 3447	1.32 6236	1.29 0827	1.25 7130	48
49	1.36 6919	1.32 9224	1.29 3391	1.25 9324	49
50	1.37 0247	1.33 2078	1.29 5831	1.26 1404	50
60	1.39 6979	1.35 4560	1.31 4668	1.27 7134	60
70	1.41 4980	1.36 9114	1.32 6379	1.28 6517	70
80	1.42 7304	1.37 8677	1.33 3756	1.29 2178	80
90	1.43 5833	1.38 5021	1.33 8441	1.29 5617	90
100	1.44 1780	1.38 9255	1.34 1432	1.29 7715	100

Table VI—Present Value of 1 per Annum

$$a_{\overline{n}|} = (1 - v^n)/i$$

n	5½%	6%	7%	8%	n
1	0.947 8673	0.943 3962	0.934 5794	0.925 9259	**1**
2	1 846 3197	1.833 3927	1.808 0182	1.783 2647	**2**
3	2.697 9334	2.673 0119	2.624 3160	2.577 0970	**3**
4	3.505 1501	3.465 1056	3.387 2113	3.312 1268	**4**
5	4.270 2845	4.212 3638	4.100 1974	3.992 7100	**5**
6	4.995 5303	4.917 3243	4.766 5397	4.622 8797	**6**
7	5.682 9671	5.582 3814	5.389 2894	5.206 3701	**7**
8	6.334 5660	6.209 7938	5.971 2985	5.746 6389	**8**
9	6.952 1952	6.801 6923	6.515 2322	6.246 8879	**9**
10	7.537 6258	7.360 0871	7.023 5816	6.710 0814	**10**
11	8 092 5363	7.886 8746	7.498 6744	7.138 9643	**11**
12	8.618 5178	8.383 8439	7.942 6863	7.536 0780	**12**
13	9.117 0785	8.852 6830	8.357 6508	7.903 7759	**13**
14	9.589 6479	9.294 9839	8.745 4680	8.244 2370	**14**
15	10.037 5809	9.712 2490	9.107 9140	8.559 4787	**15**
16	10.462 1620	10.105 8953	9.446 6486	8.851 3692	**16**
17	10.864 6086	10.477 2597	9.763 2230	9.121 6381	**17**
18	11.246 0745	10.827 6035	10.059 0869	9.371 8871	**18**
19	11.607 6535	11.158 1165	10 335 5952	9 603 5992	**19**
20	11 950 3825	11.469 9212	10 594 0142	9 818 1474	**20**
21	12.275 2441	11.764 0766	10.835 5273	10.016 8032	**21**
22	12.583 1697	12.041 5817	11.061 2405	10.200 7437	**22**
23	12.875 0424	12.303 3790	11 272 1874	10.371 0589	**23**
24	13.151 6990	12.550 3575	11 469 3340	10.528 7583	**24**
25	13.413 9327	12.783 3562	11.653 5832	10.674 7762	**25**
26	13.662 4954	13.003 1662	11.825 7787	10 809 9780	**26**
27	13.898 0999	13.210 5341	11.986 7090	10.935 1648	**27**
28	14.121 4217	13 406 1643	12.137 1113	11.051 0785	**28**
29	14 333 1012	13.590 7210	12.277 6741	11.158 4060	**29**
30	14.533 7452	13.764 8312	12.409 0412	11.257 7833	**30**
31	14.723 9291	13.929 0860	12 531 8142	11.349 7994	**31**
32	14.904 1982	14.084 0434	12 646 5553	11.434 9994	**32**
33	15.075 0694	14.230 2296	12 753 7900	11.513 8884	**33**
34	15.237 0326	14.368 1411	12 854 0094	11.586 9337	**34**
35	15.390 5522	14.498 2464	12.947 6723	11.654 5682	**35**
36	15.536 0684	14.620 9871	13 035 2078	11.717 1928	**36**
37	15.673 9985	14.736 7803	13.117 0166	11.775 1785	**37**
38	15 804 7379	14.846 0192	13.193 4735	11 828 8690	**38**
39	15.928 6615	14.949 0747	13.264 9285	11.878 5824	**39**
40	16.046 1247	15.046 2969	13.331 7088	11.924 6133	**40**
41	16.157 4642	15.138 0159	13.394 1204	11 967 2346	**41**
42	16.262 9992	15.224 5433	13.452 4490	12 006 6987	**42**
43	16.363 0324	15.306 1729	13.506 9617	12.043 2395	**43**
44	16.457 8506	15.383 1820	13.557 9081	12 077 0736	**44**
45	16.547 7257	15.455 8321	13.605 5216	12.108 4015	**45**
46	16.632 9154	15.524 3699	13.650 0202	12.137 4088	**46**
47	16.713 6639	15.589 0282	13 691 6076	12.164 2674	**47**
48	16.790 2027	15 650 0266	13.730 4744	12.189 1365	**48**
49	16.862 7514	15.707 5723	13.766 7986	12.212 1634	**49**
50	16.931 5179	15.761 8606	13.800 7463	12.233 4846	**50**
60	17.449 8542	16.161 4277	14.039 1812	12 376 5518	**60**
70	17.753 3041	16.384 5439	14.160 3893	12.442 8196	**70**
80	17.930 9529	16 509 1308	14.222 0054	12.473 5144	**80**
90	18 034 9540	16.578 6994	14.253 3279	12.487 7320	**90**
100	18 095 8394	16.617 5462	14.269 2507	12.494 3176	**100**

Table VI—Logarithm of Present Value of 1 per Annum

$\log a_{\overline{n}|}$

n	$5\frac{1}{2}\%$	6%	7%	8%	n
1	9.97 6748	9.97 4694	9.97 0616	9.96 6576	1
2	0.26 6307	0.26 3255	0.25 7203	0.25 1216	2
3	0.43 1031	0.42 7010	0.41 9016	0.41 1131	3
4	0.54 4707	0.53 9716	0.52 9843	0.52 0107	4
5	0.63 0457	0.62 4526	0.61 2805	0.60 1268	5
6	0.69 8582	0.69 1729	0.67 8203	0.66 4913	6
7	0.75 4575	0.74 6820	0.73 1532	0.71 6535	7
8	0.80 1717	0.79 3077	0.77 6069	0.75 9414	8
9	0.84 2122	0.83 2617	0.81 3930	0.79 5664	9
10	0.87 7235	0.86 6883	0.84 6559	0.82 6728	10
11	0.90 8085	0.89 6905	0.87 4984	0.85 3635	11
12	0.93 5433	0.92 3443	0.89 9967	0.87 7145	12
13	0.95 9856	0.94 7075	0.92 2084	0.89 7835	13
14	0.98 1803	0.96 8249	0.94 1783	0.91 6150	14
15	1.00 1629	0.98 7320	0.95 9419	0.93 2447	15
16	1.01 9621	1.00 4575	0.97 5278	0.94 7010	16
17	1.03 6014	1.02 0248	0.98 9593	0.96 0073	17
18	1.05 1001	1.03 4532	1.00 2559	0.97 1827	18
19	1.06 4744	1.04 7591	1.01 4335	0.98 2434	19
20	1.07 7382	1.05 9560	1.02 5061	0.99 2030	20
21	1.08 9030	1.07 0558	1.03 4850	1.00 0729	21
22	1.09 9790	1.08 0684	1.04 3804	1.00 8632	22
23	1.10 9749	1.09 0024	1.05 2008	1.01 5823	23
24	1.11 8982	1.09 8656	1.05 9538	1.02 2377	24
25	1.12 7556	1.10 6645	1.06 6459	1.02 8359	25
26	1.13 5530	1.11 4049	1.07 2830	1.03 3825	26
27	1.14 2955	1.12 0920	1.07 8700	1.03 8825	27
28	1.14 9878	1.12 7305	1.08 4115	1.04 3405	28
29	1.15 6340	1.13 3242	1.08 9116	1.04 7602	29
30	1.16 2378	1.13 8771	1.09 3738	1.05 1453	30
31	1.16 8024	1.14 3923	1.09 8014	1.05 4988	31
32	1.17 3309	1.14 8727	1.10 1972	1.05 8236	32
33	1.17 8259	1.15 3212	1.10 5639	1.06 1222	33
34	1.18 2900	1.15 7401	1.10 9039	1.06 3969	34
35	1.18 7254	1.16 1315	1.11 2192	1.06 6496	35
36	1.19 1341	1.16 4977	1.11 5118	1.06 8824	36
37	1.19 5180	1.16 8403	1.11 7835	1.07 0967	37
38	1.19 8787	1.17 1610	1.12 0359	1.07 2943	38
39	1.20 2179	1.17 4614	1.12 2705	1.07 4765	39
40	1.20 5370	1.17 7430	1.12 4886	1.07 6444	40
41	1.20 8373	1.18 0069	1.12 6914	1.07 7994	41
42	1.21 1201	1.18 2544	1.12 8801	1.07 9424	42
43	1.21 3864	1.18 4867	1.13 0558	1.08 0743	43
44	1.21 6373	1.18 7046	1.13 2193	1.08 1962	44
45	1.21 8738	1.18 9092	1.13 3715	1.08 3087	45
46	1.22 0968	1.19 1014	1.13 5133	1.08 4126	46
47	1.22 3072	1.19 2819	1.13 6454	1.08 5086	47
48	1.22 5056	1.19 4515	1.13 7686	1.08 5973	48
49	1.22 6928	1.19 6109	1.13 8833	1.08 6793	49
50	1.22 8696	1.19 7607	1.13 9903	1.08 7550	50
60	1.24 1792	1.20 8480	1.14 7342	1.09 2600	60
70	1.24 9279	1.21 4434	1.15 1075	1.09 4919	70
80	1.25 3603	1.21 7724	1.15 2961	1.09 5989	80
90	1.25 6115	1.21 9550	1.15 3916	1.09 6484	90
100	1.25 7579	1.22 0567	1.15 4401	1.09 6713	100

Table VII—Annuity Which 1 Will Buy

$$a_{\overline{n}|}^{-1} = s_{\overline{n}|}^{-1} + i$$

n	½%	1%	1¼%	1½%	n
1	1.005 0000	1 010 0000	1.012 5000	1.015 0000	1
2	0.503 7531	0.507 5124	0.509 3944	0.511 2779	2
3	0.336 6722	0.340 0221	0 341 7012	0 343 3830	3
4	0.253 1328	0.256 2811	0 257 8610	0 259 4448	4
5	0.203 0100	0.206 0398	0.207 5621	0 209 0893	5
6	0.169 5955	0.172 5484	0.174 0338	0.175 5252	6
7	0.145 7285	0.148 6283	0.150 0887	0.151 5562	7
8	0.127 8289	0.130 6903	0 132 1331	0.133 5840	8
9	0.113 9074	0.116 7404	0.118 1706	0.119 6098	9
10	0.102 7706	0.105 5821	0.107 0031	0.108 4342	10
11	0.093 6590	0.096 4541	0 097 8684	0 099 2938	11
12	0.086 0664	0.088 8488	0 090 2583	0.091 6800	12
13	0.079 6422	0.082 4148	0.083 8210	0.085 2404	13
14	0.074 1361	0.076 9012	0.078 3051	0.079 7233	14
15	0.069 3644	0.072 1238	0.073 5265	0 074 9444	15
16	0.065 1894	0.067 9446	0.069 3467	0.070 7651	16
17	0.061 5058	0.064 2581	0 065 6602	0 067 0796	17
18	0.058 2317	0.060 9820	0.062 3848	0.063 8058	18
19	0.055 3025	0.058 0518	0 059 4555	0 060 8785	19
20	0 052 6665	0.055 4153	0.056 8204	0.058 2457	20
21	0.050 2816	0.053 0308	0.054 4375	0.055 8655	21
22	0.048 1138	0.050 8637	0 052 2724	0.053 7033	22
23	0.046 1347	0.048 8858	0 050 2967	0 051 7308	23
24	0.044 3206	0.047 0735	0 048 4866	0.049 9241	24
25	0.042 6519	0.045 4068	0.046 8225	0.048 2635	25
26	0.041 1116	0.043 8689	0.045 2873	0 046 7320	26
27	0.039 6856	0.042 4455	0.043 8668	0.045 3153	27
28	0.038 3617	0.041 1244	0 042 5486	0.044 0011	28
29	0.037 1291	0.039 8950	0.041 3223	0.042 7788	29
30	0.035 9789	0.038 7481	0.040 1785	0 041 6392	30
31	0.034 9030	0.037 6757	0.039 1094	0.040 5743	31
32	0.033 8945	0.036 6709	0.038 1079	0 039 5771	32
33	0.032 9473	0.035 7274	0.037 1679	0.038 6414	33
34	0.032 0559	0.034 8400	0.036 2839	0.037 7619	34
35	0.031 2155	0.034 0037	0.035 4511	0.036 9336	35
36	0.030 4219	0 033 2143	0.034 6653	0 036 1524	36
37	0.029 6714	0 032 4680	0 033 9227	0.035 4144	37
38	0.028 9604	0 031 7615	0.033 2198	0.034 7161	38
39	0.028 2861	0.031 0916	0.032 5537	0 034 0546	39
40	0.027 6455	0.030 4556	0.031 9214	0.033 4271	40
41	0.027 0363	0.029 8510	0 031 3206	0 032 8311	41
42	0.026 4562	0.029 2756	0.030 7491	0.032 2643	42
43	0.025 9032	0.028 7274	0 030 2047	0.031 7246	43
44	0.025 3754	0.028 2044	0 029 6856	0.031 2104	44
45	0.024 8712	0.027 7050	0.029 1901	0.030 7198	45
46	0.024 3889	0.027 2277	0.028 7167	0.030 2512	46
47	0 023 9273	0.026 7711	0.028 2641	0.029 8034	47
48	0.023 4850	0.026 3338	0.027 8307	0.029 3750	48
49	0 023 0609	0.025 9147	0.027 4156	0.028 9648	49
50	0.022 6538	0.025 5127	0.027 0176	0.028 5717	50
60	0 019 3328	0.022 2444	0.023 7899	0.025 3934	60
70	0.016 9666	0.019 9328	0.021 5194	0.023 1724	70
80	0 015 1970	0.018 2188	0 019 8465	0.021 5483	80
90	0.013 8253	0.016 9031	0 018 5715	0.020 3211	90
100	0.012 7319	0 015 8657	0.017 5743	0 019 3706	100

Table VII—Annuity Which 1 Will Buy

$$a_{\overline{n|}}^{-1} = s_{\overline{n|}}^{-1} + i$$

n	1¾%	2%	2½%	3%	n
1	1.017 5000	1.020 0000	1.025 0000	1.030 0000	1
2	0.513 1629	0.515 0495	0.518 8272	0.522 6108	2
3	0.345 0675	0.346 7547	0.350 1372	0.353 5304	3
4	0.261 0324	0.262 6238	0.265 8179	0.269 0270	4
5	0.210 6214	0.212 1584	0.215 2469	0.218 3546	5
6	0.177 0226	0.178 5258	0.181 5500	0.184 5975	6
7	0.153 0306	0.154 5120	0.157 4954	0.160 5064	7
8	0.135 0429	0.136 5098	0.139 4673	0.142 4564	8
9	0.121 0581	0.122 5154	0.125 4569	0.128 4339	9
10	0.109 8754	0.111 3265	0.114 2588	0.117 2305	10
11	0.100 7304	0.102 1779	0.105 1060	0.108 0774	11
12	0.093 1138	0.094 5596	0.097 4871	0.100 4621	12
13	0.086 6728	0.088 1184	0.091 0483	0.094 0295	13
14	0.081 1556	0.082 6020	0.085 5365	0.088 5263	14
15	0.076 3774	0.077 8255	0.080 7665	0.083 7666	15
16	0.072 1996	0.073 6501	0.076 5990	0.079 6108	16
17	0.068 5162	0.069 9698	0.072 9278	0.075 9525	17
18	0.065 2449	0.066 7021	0.069 6701	0.072 7087	18
19	0.062 3206	0.063 7818	0.066 7606	0.069 8139	19
20	0.059 6912	0.061 1567	0.064 1471	0.067 2157	20
21	0.057 3146	0.058 7848	0.061 7873	0.064 8718	21
22	0.055 1564	0.056 6314	0.059 6466	0.062 7474	22
23	0.053 1880	0.054 6681	0.057 6964	0.060 8139	23
24	0.051 3857	0.052 8711	0.055 9128	0.059 0474	24
25	0.049 7295	0.051 2204	0.054 2759	0.057 4279	25
26	0.048 2027	0.049 6992	0.052 7687	0.055 9383	26
27	0.046 7908	0.048 2931	0.051 3769	0.054 5642	27
28	0.045 4815	0.046 9897	0.050 0879	0.053 2932	28
29	0.044 2642	0.045 7784	0.048 8913	0.052 1147	29
30	0.043 1298	0.044 6499	0.047 7776	0.051 0193	30
31	0.042 0701	0.043 5963	0.046 7390	0.049 9989	31
32	0.041 0781	0.042 6106	0.045 7683	0.049 0466	32
33	0.040 1478	0.041 6865	0.044 8594	0.048 1561	33
34	0.039 2736	0.040 8187	0.044 0068	0.047 3220	34
35	0.038 4508	0.040 0022	0.043 2056	0.046 5393	35
36	0.037 6751	0.039 2329	0.042 4516	0.045 8038	36
37	0.036 9426	0.038 5068	0.041 7409	0.045 1116	37
38	0.036 2499	0.037 8206	0.041 0701	0.044 4593	38
39	0.035 5940	0.037 1711	0.040 4362	0.043 8439	39
40	0.034 9721	0.036 5557	0.039 8362	0.043 2624	40
41	0.034 3817	0.035 9719	0.039 2679	0.042 7124	41
42	0.033 8206	0.035 4173	0.038 7288	0.042 1917	42
43	0.033 2867	0.034 8899	0.038 2169	0.041 6981	43
44	0.032 7781	0.034 3879	0.037 7304	0.041 2298	44
45	0.032 2932	0.033 9096	0.037 2675	0.040 7852	45
46	0.031 8304	0.033 4534	0.036 8268	0.040 3625	46
47	0.031 3884	0.033 0179	0.036 4067	0.039 9605	47
48	0.030 9657	0.032 6018	0.036 0060	0.039 5778	48
49	0.030 5612	0.032 2040	0.035 6235	0.039 2131	49
50	0.030 1739	0.031 8232	0.035 2581	0.038 8655	50
60	0.027 0534	0.028 7680	0.032 3534	0.036 1330	60
70	0.024 8893	0.026 6676	0.030 3971	0.034 3366	70
80	0.023 3209	0.025 1607	0.029 0260	0.033 1117	80
90	0.022 1476	0.024 0460	0.028 0381	0.032 2556	90
100	0.021 2488	0.023 2027	0.027 3119	0.031 6467	100

Table VII—Annuity Which 1 Will Buy

$$a_{\overline{n|}}^{-1} = s_{\overline{n|}}^{-1} + i$$

n	3½%	4%	4½%	5%	n
1	1.035 0000	1.040 0000	1.045 0000	1.050 0000	1
2	0.526 4005	0.530 1961	0.533 9976	0.537 8049	2
3	0 356 9342	0.360 3485	0.363 7734	0.367 2086	3
4	0.272 2511	0.275 4900	0.278 7436	0.282 0118	4
5	0.221 4814	0.224 6271	0.227 7916	0.230 9748	5
6	0.187 6682	0.190 7619	0.193 8784	0.197 0175	6
7	0.163 5445	0.166 6096	0.169 7015	0.172 8198	7
8	0.145 4766	0.148 5278	0.151 6097	0.154 7218	8
9	0.131 4460	0.134 4930	0.137 5745	0.140 6901	9
10	0.120 2414	0.123 2909	0.126 3788	0.129 5046	10
11	0.111 0920	0.114 1490	0.117 2482	0.120 3889	11
12	0.103 4839	0.106 5522	0.109 6662	0.112 8254	12
13	0.097 0616	0.100 1437	0.103 2754	0.106 4558	13
14	0 091 5707	0.094 6690	0.097 8203	0.101 0240	14
15	0.086 8251	0.089 9411	0.093 1138	0.096 3423	15
16	0.082 6848	0.085 8200	0.089 0154	0.092 2699	16
17	0.079 0431	0.082 1985	0 085 4176	0.088 6991	17
18	0.075 8168	0.078 9933	0.082 2369	0.085 5462	18
19	0 072 9403	0 076 1386	0.079 4073	0.082 7450	19
20	0.070 3611	0.073 5818	0.076 8761	0.080 2426	20
21	0 068 0366	0 071 2801	0.074 6006	0.077 9961	21
22	0.065 9321	0.069 1988	0.072 5456	0.075 9705	22
23	0.064 0188	0.067 3091	0.070 6825	0.074 1368	23
24	0.062 2728	0.065 5868	0.068 9870	0.072 4709	24
25	0.060 6740	0.064 0120	0.067 4390	0.070 9525	25
26	0.059 2054	0 062 5674	0.066 0214	0.069 5643	26
27	0.057 8524	0.061 2385	0.064 7195	0.068 2919	27
28	0.056 6026	0.060 0130	0.063 5208	0.067 1225	28
29	0.055 4454	0.058 8799	0.062 4146	0.066 0455	29
30	0.054 3713	0.057 8301	0.061 3915	0.065 0514	30
31	0.053 3724	0.056 8554	0.060 4434	0.064 1321	31
32	0.052 4415	0.055 9486	0.059 5632	0.063 2804	32
33	0.051 5724	0.055 1036	0.058 7445	0.062 4900	33
34	0.050 7597	0.054 3148	0 057 9819	0.061 7554	34
35	0.049 9983	0.053 5773	0.057 2704	0.061 0717	35
36	0.049 2842	0.052 8869	0.056 6058	0.060 4345	36
37	0.048 6132	0.052 2396	0.055 9840	0.059 8398	37
38	0.047 9821	0.051 6319	0.055 4017	0.059 2842	38
39	0.047 3878	0.051 0608	0 054 8557	0.058 7646	39
40	0.046 8273	0.050 5235	0.054 3431	0.058 2782	40
41	0.046 2982	0.050 0174	0.053 8616	0.057 8223	41
42	0.045 7983	0.049 5402	0.053 4087	0.057 3947	42
43	0.045 3254	0.049 0899	0 052 9823	0.056 9933	43
44	0.044 8777	0.048 6645	0.052 5807	0.056 6163	44
45	0.044 4534	0.048 2625	0.052 2020	0.056 2617	45
46	0.044 0511	0.047 8820	0.051 8447	0.055 9282	46
47	0.043 6692	0.047 5219	0 051 5073	0.055 6142	47
48	0 043 3065	0.047 1806	0.051 1886	0.055 3184	48
49	0 042 9617	0.046 8571	0.050 8872	0.055 0396	49
50	0.042 6337	0.046 5502	0.050 6021	0.054 7767	50
60	0.040 0886	0.044 2018	0.048 4543	0.052 8282	60
70	0 038 4610	0.042 7451	0.047 1651	0.051 6992	70
80	0.037 3849	0.041 8141	0.046 3707	0.051 0296	80
90	0.036 6578	0.041 2078	0.045 8732	0.050 6271	90
100	0.036 1593	0.040 8080	0 045 5584	0 050 3831	100

Table VII—Annuity Which 1 Will Buy

$$a_{\overline{n}|}^{-1} = s_{\overline{n}|}^{-1} + i$$

n	$5\frac{1}{2}\%$	6%	7%	8%	n
1	1.055 0000	1.060 0000	1.070 0000	1.080 0000	1
2	0.541 6180	0.545 4369	0.553 0918	0.560 7692	2
3	0.370 6541	0.374 1098	0.381 0517	0.388 0335	3
4	0.285 2945	0.288 5915	0.295 2281	0.301 9208	4
5	0.234 1764	0.237 3964	0.243 8907	0.250 4565	5
6	0.200 1789	0.203 3626	0.209 7958	0.216 3154	6
7	0.175 9644	0.179 1350	0.185 5532	0.192 0724	7
8	0.157 8640	0.161 0359	0.167 4678	0.174 0148	8
9	0.143 8395	0.147 0222	0.153 4865	0.160 0797	9
10	0.132 6678	0.135 8680	0.142 3775	0.149 0295	10
11	0.123 5707	0.126 7929	0.133 3569	0.140 0763	11
12	0.116 0292	0.119 2770	0.125 9020	0.132 6950	12
13	0.109 6843	0.112 9601	0.119 6508	0.126 5218	13
14	0.104 2791	0.107 5849	0.114 3449	0.121 2969	14
15	0.099 6256	0.102 9628	0.109 7946	0.116 8295	15
16	0.095 5825	0.098 9521	0.105 8576	0.112 9769	16
17	0.092 0420	0.095 4448	0.102 4252	0.109 6294	17
18	0.088 9199	0.092 3565	0.099 4126	0.106 7021	18
19	0.086 1501	0.089 6209	0.096 7530	0.104 1276	19
20	0.083 6793	0.087 1846	0.094 3929	0.101 8522	20
21	0.081 4648	0.085 0045	0.092 2890	0.099 8323	21
22	0.079 4712	0.083 0456	0.090 4058	0.098 0321	22
23	0.077 6696	0.081 2785	0.088 7139	0.096 4222	23
24	0.076 0358	0.079 6790	0.087 1890	0.094 9780	24
25	0.074 5494	0.078 2267	0.085 8105	0.093 6788	25
26	0.073 1931	0.076 9043	0.084 5610	0.092 5071	26
27	0.071 9523	0.075 6972	0.083 4257	0.091 4481	27
28	0.070 8144	0.074 5926	0.082 3919	0.090 4889	28
29	0.069 7686	0.073 5796	0.081 4486	0.089 6185	29
30	0.068 8054	0.072 6489	0.080 5864	0.088 8274	30
31	0.067 9167	0.071 7922	0.079 7969	0.088 1073	31
32	0.067 0952	0.071 0023	0.079 0729	0.087 4508	32
33	0.066 3347	0.070 2729	0.078 4081	0.086 8516	33
34	0.065 6296	0.069 5984	0.077 7967	0.086 3041	34
35	0.064 9749	0.068 9739	0.077 2340	0.085 8033	35
36	0.064 3663	0.068 3948	0.076 7153	0.085 3447	36
37	0.063 7999	0.067 8574	0.076 2368	0.084 9244	37
38	0.063 2722	0.067 3581	0.075 7951	0.084 5389	38
39	0.062 7799	0.066 8938	0.075 3868	0.084 1851	39
40	0.062 3203	0.066 4615	0.075 0091	0.083 8602	40
41	0.061 8909	0.066 0589	0.074 6596	0.083 5615	41
42	0.061 4893	0.065 6834	0.074 3359	0.083 2868	42
43	0.061 1134	0.065 3331	0.074 0359	0.083 0341	43
44	0.060 7613	0.065 0061	0.073 7577	0.082 8015	44
45	0.060 4313	0.064 7005	0.073 4996	0.082 5873	45
46	0.060 1218	0.064 4149	0.073 2600	0.082 3899	46
47	0.059 8313	0.064 1477	0.073 0374	0.082 2080	47
48	0.059 5585	0.063 8977	0.072 8307	0.082 0403	48
49	0.059 3023	0.063 6636	0.072 6385	0.081 8856	49
50	0.059 0615	0.063 4443	0.072 4598	0.081 7429	50
60	0.057 3071	0.061 8757	0.071 2292	0.080 7979	60
70	0.056 3275	0.061 0331	0.070 6195	0.080 3676	70
80	0.055 7695	0.060 5725	0.070 3136	0.080 1699	80
90	0.055 4479	0.060 3184	0.070 1591	0.080 0786	90
100	0.055 2613	0.060 1774	0.070 0808	0.080 0364	100

Table VIII—Amount of 1 for Fractional Periods

$(1 + i)^{1/p}$

p	½%	1%	1¼%	1½%	p
2	1.002 4969	1.004 9876	1.006 2306	1.007 4721	2
3	1.001 6639	1.003 3223	1.004 1494	1.004 9752	3
4	1.001 2477	1.002 4907	1.003 1105	1.003 7291	4
6	1.000 8316	1.001 6598	1.002 0726	1.002 4845	6
12	1.000 4157	1.000 8295	1.001 0357	1.001 2415	12
p	1¾%	2%	2½%	3%	p
2	1.008 7121	1.009 9505	1.012 4228	1.014 8892	2
3	1.005 7996	1.006 6227	1.008 2648	1.009 9016	3
4	1.004 3466	1.004 9629	1.006 1922	1.007 4171	4
6	1.002 8956	1.003 3059	1.004 1239	1.004 9386	6
12	1.001 4468	1.001 6516	1.002 0598	1.002 4663	12
p	3½%	4%	4½%	5%	p
2	1.017 3495	1.019 8039	1.022 2524	1.024 6951	2
3	1.011 5331	1.013 1594	1.014 7805	1.016 3964	3
4	1.008 6374	1.009 8534	1.011 0650	1.012 2722	4
6	1.005 7500	1.006 5582	1.007 3631	1.008 1648	6
12	1.002 8709	1.003 2737	1.003 6748	1 004 0741	12
p	5½%	6%	7%	8%	p
2	1.027 1319	1.029 5630	1.034 4080	1.039 2305	2
3	1.018 0071	1.019 6128	1.022 8091	1.025 9856	3
4	1.013 4752	1.014 6738	1.017 0585	1.019 4265	4
6	1.008 9634	1.009 7588	1.011 3403	1.012 9095	6
12	1.004 4717	1.004 8676	1.005 6541	1.006 4340	12

Logarithm of Amount of 1 for Fractional Periods

$\log (1 + i)^{1/p}$

p	½%	1%	1¼%	1½%	p
2	0.00 1083	0.00 2161	0.00 2698	0.00 3233	2
3	0.00 0722	0.00 1440	0.00 1798	0.00 2155	3
4	0.00 0542	0.00 1080	0.00 1349	0.00 1617	4
6	0.00 0361	0.00 0720	0.00 0899	0.00 1078	6
12	0.00 0180	0.00 0360	0.00 0450	0.00 0539	12
p	1¾%	2%	2½%	3%	p
2	0.00 3767	0.00 4300	0.00 5362	0.00 6419	2
3	0.00 2511	0.00 2867	0.00 3575	0 00 4279	3
4	0.00 1884	0.00 2150	0.00 2681	0.00 3209	4
6	0.00 1256	0.00 1433	0 00 1787	0.00 2140	6
12	0.00 0628	0 00 0717	0.00 0894	0.00 1070	12
p	3½%	4%	4½%	5%	p
2	0.00 7470	0.00 8517	0.00 9558	0.01 0595	2
3	0.00 4980	0.00 5678	0.00 6372	0.00 7063	3
4	0.00 3735	0.00 4258	0.00 4779	0 00 5297	4
6	0.00 2490	0.00 2839	0.00 3186	0.00 3532	6
12	0.00 1245	0.00 1419	0.00 1593	0.00 1766	12
p	5½%	6%	7%	8%	p
2	0.01 1626	0.01 2653	0.01 4692	0.01 6712	2
3	0.00 7751	0.00 8435	0.00 9795	0 01 1141	3
4	0.00 5813	0.00 6326	0.00 7346	0.00 8356	4
6	0.00 3875	0.00 4218	0.00 4897	0.00 5571	6
12	0.00 1938	0.00 2109	0.00 2449	0 00 2785	12

Table IX—Nominal Rate of Interest *j* Converted *p* Times Corresponding to Effective Rate *i*

$$j_{(p)} = p[(1+i)^{1/p} - 1]$$

p	½%	1%	1¼%	1½%	p
2	.004 9938	.009 9751	.012 4612	.014 9442	2
3	.004 9917	.009 9669	.012 4483	.014 9256	3
4	.004 9907	.009 9627	.012 4418	.014 9164	4
6	.004 9896	.009 9586	.012 4354	.014 9071	6
12	.004 9886	.009 9545	.012 4290	.014 8979	12

p	1¾%	2%	2½%	3%	p
2	.017 4241	.019 9010	.024 8457	.029 7783	2
3	.017 3989	.019 8681	.024 7945	.029 7049	3
4	.017 3863	.019 8517	.024 7690	.029 6683	4
6	.017 3737	.019 8353	.024 7435	.029 6317	6
12	.017 3612	.019 8190	.024 7180	.029 5952	12

p	3½%	4%	4½%	5%	p
2	.034 6990	.039 6078	.044 5048	.049 3902	2
3	.034 5994	.039 4782	.044 3414	.049 1891	3
4	.034 5498	.039 4136	.044 2600	.049 0889	4
6	.034 5002	.039 3492	.044 1787	.048 9891	6
12	.034 4508	.039 2849	.044 0977	.048 8895	12

p	5½%	6%	7%	8%	p
2	.054 2639	.059 1260	.068 8161	.078 4610	2
3	.054 0214	.058 8385	.068 4274	.077 9567	3
4	.053 9007	.058 6954	.068 2341	.077 7062	4
6	.053 7804	.058 5528	.068 0416	.077 4567	6
12	.053 6604	.058 4106	.067 8497	.077 2084	12

Logarithm of Nominal Rate of Interest *j* Converted *p* Times Corresponding to Effective Rate *i*

$$\log j_{(p)}$$

p	½%	1%	1¼%	1½%	p
2	7.69 8428	7.99 8918	8.09 5559	8.17 4472	2
3	7.69 8248	7.99 8558	8.09 5109	8.17 3932	3
4	7.69 8157	7.99 8378	8.09 4884	8.17 3663	4
6	7.69 8067	7.99 8198	8.09 4659	8.17 3393	6
12	7.69 7977	7.99 8017	8.09 4434	8.17 3124	12

p	1¾%	2%	2½%	3%	p
2	8.24 1150	8.29 8875	8.39 5250	8.47 3900	2
3	8.24 0522	8.29 8157	8.39 4356	8.47 2828	3
4	8.24 0207	8.29 7798	8.39 3908	8.47 2293	4
6	8.23 9893	8.29 7440	8.39 3461	8.47 1757	6
12	8.23 9579	8.29 7081	8.39 3014	8.47 1222	12

p	3½%	4%	4½%	5%	p
2	8.54 0317	8.59 7781	8.64 8407	8.69 3640	2
3	8.53 9069	8.59 6357	8.64 6809	8.69 1869	3
4	8.53 8445	8.59 5646	8.64 6011	8.69 0984	4
6	8.53 7822	8.59 4936	8.64 5213	8.69 0099	6
12	8.53 7199	8.59 4225	8.64 4416	8.68 9216	12

p	5½%	6%	7%	8%	p
2	8.73 4511	8.77 1779	8.83 7690	8.89 4654	2
3	8.73 2566	8.76 9661	8.83 5230	8.89 1853	3
4	8.73 1594	8.76 8604	8.83 4001	8.89 0456	4
6	8.73 0624	8.76 7547	8.83 2774	8.88 9059	6
12	8.72 9654	8.76 6492	8.83 1548	8.88 7664	12

Table X—Amount at End of Year of p Deposits Each of $1/p$ Deposited at End of Each pth Part of a Year

$$s_{\overline{1}|}^{(p)} = i/j_{(p)}$$

p	½%	1%	1¼%	1½%	p
2	1.001 2484	1.002 4938	1.003 1153	1.003 7360	2
3	1.001 6648	1.003 3260	1.004 1552	1.004 9835	3
4	1.001 8731	1.003 7422	1.004 6754	1.005 6076	4
6	1.002 0813	1.004 1586	1.005 1958	1.006 2319	6
12	1.002 2896	1.004 5751	1.005 7163	1.006 8565	12
p	1¾%	2%	2½%	3%	p
2	1.004 3618	1.004 9752	1.006 2114	1.007 4446	2
3	1.005 8108	1.006 6373	1.008 2876	1.009 9343	3
4	1.006 5388	1.007 4686	1.009 3268	1.011 1807	4
6	1.007 2671	1.008 3012	1.010 3667	1.012 4282	6
12	1.007 9957	1.009 1339	1.011 4072	1.013 6766	12
p	3½%	4%	4½%	5%	p
2	1.008 6748	1.009 9020	1.011 1262	1.012 3475	2
3	1.011 5775	1.013 2171	1.014 8533	1.016 4860	3
4	1.013 0309	1.014 8774	1.016 7203	1.018 5594	4
6	1.014 4858	1.016 5396	1.018 5895	1.020 6357	6
12	1.015 9420	1.018 2035	1.020 4611	1.022 7148	12
p	5½%	6%	7%	8%	p
2	1.013 5660	1.014 7815	1.017 2040	1.019 6152	2
3	1.018 1152	1.019 7410	1.022 9825	1.026 2106	3
4	1.020 3950	1.022 2269	1.025 8800	1.029 5190	4
6	1.022 6781	1.024 7168	1.028 7830	1.032 8346	6
12	1.024 9647	1.027 2107	1.031 6914	1.036 1572	12

Logarithm of Amount at End of Year of p Deposits Each of $1/p$ Deposited at End of Each pth Part of a Year

$$\log s_{\overline{1}|}^{(p)}$$

p	½%	1%	1¼%	1½%	p
2	0.00 0542	0.00 1082	0.00 1351	0.00 1620	2
3	0.00 0722	0.00 1442	0.00 1801	0.00 2159	3
4	0.00 0813	0.00 1622	0.00 2026	0.00 2429	4
6	0.00 0903	0.00 1802	0.00 2251	0.00 2698	6
12	0.00 0993	0.00 1982	0.00 2475	0.00 2968	12
p	1¾%	2%	2½%	3%	p
2	0.00 1888	0.00 2155	0.00 2689	0.00 3221	2
3	0.00 2516	0.00 2873	0.00 3584	0.00 4293	3
4	0.00 2831	0.00 3232	0.00 4032	0.00 4829	4
6	0.00 3145	0.00 3590	0.00 4479	0.00 5364	6
12	0.00 3459	0.00 3949	0.00 4926	0.00 5899	12
p	3½%	4%	4½%	5%	p
2	0.00 3751	0.00 4279	0.00 4805	0.00 5330	2
3	0.00 4999	0.00 5703	0.00 6403	0.00 7101	3
4	0.00 5623	0.00 6414	0.00 7201	0.00 7986	4
6	0.00 6246	0.00 7124	0.00 7999	0.00 8871	6
12	0.00 6869	0.00 7835	0.00 8796	0.00 9755	12
p	5½%	6%	7%	8%	p
2	0.00 5852	0.00 6373	0.00 7408	0.00 8436	2
3	0.00 7797	0.00 8490	0.00 9868	0.01 1236	3
4	0.00 8768	0.00 9547	0.01 1097	0.01 2634	4
6	0.00 9739	0.01 0604	0.01 2324	0.01 4031	6
12	0.01 0709	0.01 1660	0.01 3550	0.01 5426	12

Table XI—American Experience Table of Mortality

Age x	Number Living l_x	Number of Deaths d_x	Yearly Probability of Dying q_x	Yearly Probability of Living p_x	Age x	Number Living l_x	Number of Deaths d_x	Yearly Probability of Dying q_x	Yearly Probability of Living p_x
10	100,000	749	0.007 490	0.992 510	**53**	66,797	1,091	0.016 333	0.983 667
11	99,251	746	0.007 516	0.992 484	**54**	65,706	1,143	0.017 396	0.982 604
12	98,505	743	0.007 543	0.992 457	**55**	64,563	1,199	0 018 571	0.981 429
13	97,762	740	0.007 569	0 992 431	**56**	63,364	1,260	0.019 885	0.980 115
14	97,022	737	0.007 596	0.992 404	**57**	62,104	1,325	0.021 335	0.978 665
15	96,285	735	0.007 634	0 992 366	**58**	60,779	1,394	0.022 936	0.977 064
16	95,550	732	0.007 661	0.992 339	**59**	59,385	1,468	0.024 720	0.975 280
17	94,818	729	0.007 688	0.992 312	**60**	57,917	1,546	0.026 693	0.973 307
18	94,089	727	0.007 727	0.992 273	**61**	56,371	1,628	0.028 880	0.971 120
19	93,362	725	0.007 765	0.992 235	**62**	54,743	1,713	0.031 292	0.968 708
20	92,637	723	0.007 805	0.992 195	**63**	53,030	1,800	0.033 943	0.966 057
21	91,914	722	0.007 855	0.992 145	**64**	51,230	1,889	0.036 873	0.963 127
22	91,192	721	0.007 906	0.992 094	**65**	49,341	1,980	0.040 129	0 959 871
23	90,471	720	0.007 958	0 992 042	**66**	47,361	2,070	0.043 707	0.956 293
24	89,751	719	0.008 011	0.991 989	**67**	45,291	2,158	0.047 647	0 952 353
25	89,032	718	0.008 065	0.991 935	**68**	43,133	2,243	0.052 002	0.947 998
26	88,314	718	0.008 130	0.991 870	**69**	40,890	2,321	0.056 762	0.943 238
27	87,596	718	0.008 197	0.991 803	**70**	38,569	2,391	0.061 993	0 938 007
28	86,878	718	0.008 264	0.991 736	**71**	36,178	2,448	0.067 665	0.932 335
29	86,160	719	0.008 345	0.991 655	**72**	33,730	2,487	0.073 733	0 926 267
30	85,441	720	0 008 427	0.991 573	**73**	31,243	2,505	0.080 178	0 919 822
31	84,721	721	0 008 510	0.991 490	**74**	28,738	2,501	0.087 028	0 912 972
32	84,000	723	0.008 607	0.991 393	**75**	26,237	2,476	0.094 371	0 905 629
33	83,277	726	0.008 718	0.991 282	**76**	23,761	2,431	0.102 311	0.897 689
34	82,551	729	0.008 831	0.991 169	**77**	21,330	2,369	0.111 064	0.888 936
35	81,822	732	0 008 946	0.991 054	**78**	18,961	2,291	0.120 827	0 879 173
36	81,090	737	0 009 089	0.990 911	**79**	16,670	2,196	0.131 734	0.868 260
37	80,353	742	0.009 234	0.990 766	**80**	14,474	2,091	0.144 466	0 855 534
38	79,611	749	0.009 408	0.990 592	**81**	12,383	1,964	0.158 605	0 841 395
39	78,862	756	0.009 586	0.990 414	**82**	10,419	1,816	0.174 297	0.825 703
40	78,106	765	0.009 794	0.990 206	**83**	8,603	1,648	0.191 561	0.808 439
41	77,341	774	0.010 008	0.989 992	**84**	6,955	1,470	0.211 359	0.788 641
42	76,567	785	0.010 252	0.989 748	**85**	5,485	1,292	0.235 552	0.764 448
43	75,782	797	0 010 517	0.989 483	**86**	4,193	1,114	0.265 681	0.734 319
44	74,985	812	0.010 829	0.989 171	**87**	3,079	933	0.303 020	0.696 980
45	74,173	828	0.011 163	0.988 837	**88**	2,146	744	0.346 692	0.653 308
46	73,345	848	0.011 562	0.988 438	**89**	1,402	555	0.395 863	0.604 137
47	72,497	870	0.012 000	0.988 000	**90**	847	385	0.454 545	0.545 455
48	71,627	896	0 012 509	0.987 491	**91**	462	246	0.532 468	0.467 534
49	70,731	927	0.013 100	0.986 894	**92**	216	137	0.634 259	0.365 741
50	69,804	962	0.013 781	0.986 219	**93**	79	58	0.734 177	0.265 823
51	68,842	1,001	0.014 541	0.985 459	**94**	21	18	0.857 143	0.142 857
52	67,841	1,044	0.015 389	0.984 611	**95**	3	3	1.000 000	0 000 000

Table XII—Commutation Columns

American Experience, 3%

x	D_x	N_x	C_x	M_x	x
10	74 409.4	1 811 346	541.094	21 651 7	**10**
11	71 701.0	1 736 936	523.229	21 110.7	**11**
12	69 089.4	1 665 235	505.947	20 587.4	**12**
13	66 571.2	1 596 146	489.227	20 081.5	**13**
14	64 143.0	1 529 575	473.052	19 592.3	**14**
15	61 801.7	1 465 432	458.028	19 119.2	**15**
16	59 543.6	1 403 630	442 872	18 661.2	**16**
17	57 366.4	1 344 086	428.211	18 218 3	**17**
18	55 267.4	1 286 720	414 598	17 790 1	**18**
19	53 243.0	1 231 453	401.415	17 375 5	**19**
20	51 290.9	1 178 210	388.648	16 974 1	**20**
21	49 408.3	1 126 919	376 806	16 585 4	**21**
22	47 592.4	1 077 510	365.325	16 208.6	**22**
23	45 840.9	1 029 918	354.192	15 843 3	**23**
24	44 151.5	984 077	343.398	15 489.1	**24**
25	42 522.2	939 926	332.933	15 145.7	**25**
26	40 950.7	897 403	323.236	14 812.8	**26**
27	39 434.8	856 453	313 821	14 489 5	**27**
28	37 972.4	817 018	304.681	14 175 7	**28**
29	36 561.7	779 046	296.218	13 871.0	**29**
30	35 200.6	742 484	287.991	13 574.8	**30**
31	33 887.3	707 283	279 991	13 286 8	**31**
32	32 620.3	673 396	272.590	13 006.8	**32**
33	31 397.6	640 776	265.749	12 734.2	**33**
34	30 217.4	609 378	259.074	12 468.5	**34**
35	29 078.2	579 161	252.564	12 209.4	**35**
36	27 978.7	550 082	246 882	11 956.9	**36**
37	26 916.9	522 104	241.318	11 710 0	**37**
38	25 891.6	495 187	236.499	11 468 7	**38**
39	24 901.0	469 295	231.757	11 232.2	**39**
40	23 943.9	444 394	227.685	11 000.4	**40**
41	23 018.8	420 450	223.654	10 772.7	**41**
42	22 124.7	397 432	220.226	10 549.1	**42**
43	21 260.1	375 307	217.080	10 328.8	**43**
44	20 423.8	354 047	214.724	10 111.8	**44**
45	19 614.2	333 623	212.578	9 897.03	**45**
46	18 830.3	314 009	211.371	9 684.45	**46**
47	18 070.5	295 178	210.539	9 473.08	**47**
48	17 333.6	277 108	210.515	9 262 54	**48**
49	16 618.3	259 774	211.455	9 052.03	**49**
50	15 922.8	243 156	213.048	8 840 57	**50**
51	15 246.0	227 233	215.228	8 627.53	**51**
52	14 586.7	211 987	217.935	8 412.30	**52**
53	13 943.9	197 401	221.113	8 194 36	**53**
54	13 316.6	183 457	224.905	7 973 25	**54**
55	12 703.9	170 140	229.052	7 748 34	**55**
56	12 104.8	157 436	233.695	7 519.29	**56**
57	11 518.5	145 331	238.593	7 285 60	**57**
58	10 944.5	133 813	243.706	7 047.00	**58**
59	10 382.0	122 868	249.168	6 803.30	**59**
60	9 830.43	112 486	254.764	6 554.13	**60**
61	9 289.34	102 656	260.463	6 299.37	**61**
62	8 758.32	93 366.6	266.080	6 038.90	**62**
63	8 237.14	84 608.2	271.450	5 772 82	**63**
64	7 725.77	76 371.1	276.575	5 501.37	**64**

Table XII—Commutation Columns

American Experience, 3%

x	D_x	N_x	C_x	M_x	x
65	7 224.18	68 645 3	281.455	5 224 80	65
66	6 732.31	61 421.2	285 678	4 943 34	66
67	6 250.54	54 688 8	289 148	4 657.67	67
68	5 779.34	48 438 3	291.784	4 368.52	68
69	5 319.23	42 659.0	293.136	4 076.73	69
70	4 871.16	37 339.7	293.182	3 783.60	70
71	4 436 10	32 468.6	291 428	3 490 42	71
72	4 015.47	28 032.5	287.447	3 198.99	72
73	3 611.07	24 017 0	281 095	2 911.54	73
74	3 224 79	20 405 9	272.472	2 630.45	74
75	2 858.40	17 181 1	261.892	2 357.97	75
76	2 513.25	14 322 7	249 643	2 096 08	76
77	2 190.41	11 809 5	236.190	1 846 44	77
78	1 890 42	9 619 09	221.761	1 610 25	78
79	1 613.60	7 728 67	206.374	1 388.49	79
80	1 360 22	6 115 07	190.783	1 182 12	80
81	1 129 82	4 754 85	173 976	991 333	81
82	922 940	3 625.02	156.180	817 357	82
83	739.878	2 702 08	137.604	661.177	83
84	580.725	1 962.21	119.166	523.573	84
85	444.644	1 381 48	101.686	404.407	85
86	330 007	936 837	85.122 9	302 721	86
87	235 272	606 829	69 215 9	217.598	87
88	159.204	371 557	53 587 1	148.382	88
89	100.980	212.353	38 809 9	94.794 9	89
90	59.228 8	111 373	26.138 1	55.985 0	90
91	31 365 7	52.144 2	16.214 8	29 846 9	91
92	14.237 3	20 778 5	8.767 15	13.632 1	92
93	5.055 51	6.541 20	3 603 54	4.864 99	93
94	1.304 73	1.485 69	1.085 77	1.261 46	94
95	0.180 961	0.180 961	0.175 690	0.175 690	95

Table XIII—1937 Standard Annuity Mortality Table

Age		l_x	Age		l_x
Male	Female		Male	Female	
5	10	1,000,000	60	65	769,853
6	11	998,766	61	66	754,646
7	12	997,525	62	67	738,574
8	13	996,278	63	68	721,618
9	14	995,028	64	69	703,759
10	15	993,778	65	70	684,986
11	16	992,529	66	71	665,292
12	17	991,282	67	72	644,677
13	18	990,036	68	73	623,151
14	19	988,792	69	74	600,731
15	20	987,547	70	75	577,445
16	21	986,301	71	76	553,332
17	22	985,051	72	77	528,443
18	23	983,793	73	78	502,843
19	24	982,524	74	79	476,611
20	25	981,239	75	80	449,841
21	26	979,933	76	81	422,642
22	27	978,600	77	82	395,136
23	28	977,232	78	83	367,464
24	29	975,823	79	84	339,776
25	30	974,363	80	85	312,237
26	31	972,842	81	86	285,022
27	32	971,252	82	87	258,313
28	33	969,580	83	88	232,295
29	34	967,813	84	89	207,154
30	35	965,939	85	90	183,071
31	36	963,944	86	91	160,217
32	37	961,812	87	92	138,748
33	38	959,526	88	93	118,798
34	39	957,068	89	94	100,478
35	40	954,424	90	95	83,867
36	41	951,579	91	96	69,011
37	42	948,519	92	97	55,919
38	43	945,228	93	98	44,564
39	44	941,691	94	99	34,884
40	45	937,889	95	100	26,784
41	46	933,804	96	101	20,140
42	47	929,416	97	102	14,807
43	48	924,706	98	103	10,609
44	49	919,650	99	104	7,365
45	50	914,226	100	105	4,921
46	51	908,410	101	106	3,139
47	52	902,176	102	107	1,891
48	53	895,497	103	108	1,061
49	54	888,348	104	109	544
50	55	880,697	105	110	249
51	56	872,517	106	111	97
52	57	863,776	107	112	30
53	58	854,443	108	113	6
54	59	844,486	109	114	1
55	60	833,874	110	115	0
56	61	822,572			
57	62	810,551			
58	63	797,777			
59	64	784,221			

Table XIV—1937 Standard Annuity Table with 3% Interest Commutation Columns

Age		D_x	N_x	M_x	a_x 3%
Male	Female				
5	**10**	862 609	24 625 723	145 354.7	27 547 96
6	**11**	836 451	23 763 115	144 321 3	27 409 46
7	**12**	811 079	22 926 664	143 312 2	27.266 87
8	**13**	786 471	22 115 585	142 327.8	27.120 02
9	**14**	762 606	21 329 114	141 369.8	26.968 72
10	**15**	739 464	20 566 508	140 439.7	26 812 72
11	**16**	717 024	19 827 043	139 537 4	26 651 85
12	**17**	695 265	19 110 019	138 662 7	26.485 94
13	**18**	674 166	18 414 754	137 814 3	26 314 85
14	**19**	653 708	17 740 588	136 991.9	26.138 40
15	**20**	633 869	17 086 880	136 192 7	25 956 49
16	**21**	614 630	16 453 011	135 416 3	25 768 96
17	**22**	595 972	15 838 381	134 660 0	25 575 71
18	**23**	577 875	15 242 409	133 921.1	25 376 67
19	**24**	560 320	14 664 534	133 197.4	25 171 73
20	**25**	543 288	14 104 214	132 485 9	24 960 83
21	**26**	526 762	13 560 926	131 783 8	24 743 92
22	**27**	510 724	13 034 164	131 088.2	24 520 95
23	**28**	495 155	12 523 440	130 395 0	24 291 94
24	**29**	480 040	12 028 284	129 701.9	24 056 82
25	**30**	465 361	11 548 244	129 004 6	23.815 66
26	**31**	451 102	11 082 883	128 299.3	23 568 48
27	**32**	437 247	10 631 781	127 583.5	23 315 27
28	**33**	423 781	10 194 534	126 852.7	23.056 14
29	**34**	410 688	9 770 753	126 102.9	22.791 19
30	**35**	397 954	9 360 065	125 330.8	22 520 47
31	**36**	385 565	8 962 111	124 532 8	22 244 09
32	**37**	373 507	8 576 546	123 704 9	21.962 20
33	**38**	361 766	8 203 039	122 843 0	21.674 95
34	**39**	350 330	7 841 272	121 943.3	21.382 54
35	**40**	339 186	7 490 942	121 003.7	21 085 03
36	**41**	328 326	7 151 756	120 022 0	20.782 51
37	**42**	317 738	6 823 430	118 997 0	20 475 04
38	**43**	307 413	6 505 693	117 926 7	20 162 72
39	**44**	297 342	6 198 280	116 809.8	19.845 61
40	**45**	287 516	5 900 938	115 644 3	19 523 84
41	**46**	277 926	5 613 421	114 428.5	19.197 52
42	**47**	268 563	5 335 495	113 160 6	18.866 80
43	**48**	259 420	5 066 932	111 839 2	18.531 79
44	**49**	250 487	4 807 512	110 462 1	18.192 68
45	**50**	241 757	4 557 025	109 027.8	17.849 64
46	**51**	233 222	4 315 269	107 534.6	17.502 83
47	**52**	224 875	4 082 047	105 980.7	17 152 49
48	**53**	216 709	3 857 171	104 364.4	16 798 84
49	**54**	208 718	3 640 462	102 684.7	16.442 04
50	**55**	200 893	3 431 745	100 939.5	16.082 43
51	**56**	193 230	3 230 851	99 127.9	15 720 20
52	**57**	185 723	3 037 621	97 248 5	15.355 66
53	**58**	178 365	2 851 898	95 300.2	14.989 09
54	**59**	171 152	2 673 533	93 282.2	14.620 80
55	**60**	164 079	2 502 381	91 194.2	14.251 07
56	**61**	157 141	2 338 302	89 035.1	13 880 28
57	**62**	150 334	2 181 161	86 805.5	13.508 72
58	**63**	143 656	2 030 826	84 505 3	13.136 77
59	**64**	137 102	1 887 171	82 135.4	12.764 77

Table XIV—1937 Standard Annuity Table with 3% Interest Commutation Columns

AGE		D_x	N_x	M_x	a_x 3%
Male	Female				
60	65	130 670.	1 750 069.	79 696.6	12 393 09
61	66	124 358.	1 619 400.	77 190.7	12.022 11
62	67	118 164.	1 495 042.	74 619.3	11.652 24
63	68	112 089.	1 376 878.	71 985 6	11.283 81
64	69	106 131.	1 264 789.	69 292.3	10.917 26
65	70	100 291.	1 158 658.	66 543 7	10.552 96
66	71	94 570.5	1 058 367.	63 744 2	10.191 31
67	72	88 970.9	963 796.6	60 899.2	9.832 71
68	73	83 495.3	874 825.7	58 014.9	9.477 54
69	74	78 146.9	791 330.4	55 098.4	9.126 20
70	75	72 929.8	713 183.5	52 157.4	8.779 04
71	76	67 848.9	640 253.7	49 200.7	8.436 46
72	77	62 909.8	572 404.8	46 237 8	8.098 83
73	78	58 118.6	509 495.1	43 278 9	7.766 47
74	79	53 482.2	451 376.5	40 335.3	7.439 75
75	80	49 008.0	397 894.3	37 418.9	7.118 96
76	81	44 703.7	348 886.2	34 542.0	6.804 41
77	82	40 577.0	304 182.5	31 717.3	6.496 42
78	83	36 636.3	263 605.5	28 958.4	6.195 21
79	84	32 889.1	226 969.2	26 278.3	5.901 05
80	85	29 343.1	194 080.1	23 690.3	5.614 16
81	86	26 005.4	164 737.0	21 207 2	5.334 73
82	87	22 882.0	138 731 6	18 841.3	5.062 92
83	88	19 977.9	115 849.6	16 603.7	4.798 88
84	89	17 296.8	95 871.7	14 504.4	4.542 73
85	90	14 840.7	78 574.9	12 552.1	4.294 54
86	91	12 609.8	63 734.2	10 753.4	4.054 35
87	92	10 602.0	51 124 4	9 112.9	3.822 14
88	93	8 813.19	40 522.37	7 632.9	3.597 92
89	94	7 236.99	31 709.17	6 313.4	3.381 54
90	95	5 864.63	24 472.18	5 151.9	3.172 84
91	96	4 685.23	18 607.55	4 143 26	2.971 53
92	97	3 685.83	13 922.32	3 280 32	2.777 26
93	98	2 851.82	10 236.50	2 553.67	2.589 46
94	99	2 167.34	7 384.68	1 952.25	2.407 25
95	100	1 615.62	5 217.33	1 463 66	2.229 31
96	101	1 179.47	3 601.72	1 074.56	2.053 68
97	102	841.892	2 422.248	771.34	1.877 15
98	103	585.634	1 580.356	539.60	1.698 54
99	104	394.719	994.722	365.75	1.520 08
100	105	256.054	600.003	238.578	1.343 27
101	106	158.574	343.950	148 556	1.169 02
102	107	92.745 9	185.376	87.347	.998 75
103	108	50.522 1	92.630 1	47.824	.833 46
104	109	25.149 4	42.108 0	23.923	.674 31
105	110	11.176 1	16.958 6	10.682 2	.517 40
106	111	4.226 94	5.782 49	4.058 52	.368 01
107	112	1.269 22	1.555 55	1.223 92	.225 59
108	113	.246 451	.286 330	.238 112	.161 81
109	114	.039 879	.039 879	.038 717	.000 00
110	115				

Table XV—Six-Place Logarithms

N	Log.	*N*	Log.	*N*	Log	*N*.	Log	*N*	Log
0	-	**20**	1.30 1030	**40**	1.60 2060	**60**	1 77 8151	**80**	1 90 3090
1	0.00 0000	**21**	1.32 2219	**41**	1 61 2784	**61**	1 78 5330	**81**	1 90 8485
2	0.30 1030	**22**	1 34 2423	**42**	1 62 3249	**62**	1 79 2392	**82**	1 91 3814
3	0.47 7121	**23**	1 36 1728	**43**	1 63 3468	**63**	1 79 9341	**83**	1 91 9078
4	0.60 2060	**24**	1.38 0211	**44**	1 64 3453	**64**	1 80 6180	**84**	1 92 4279
5	0 69 8970	**25**	1 39 7940	**45**	1 65 3213	**65**	1 81 2913	**85**	1 92 9419
6	0.77 8151	**26**	1 41 4973	**46**	1 66 2758	**66**	1 81 9544	**86**	1 93 4498
7	0.84 5098	**27**	1 43 1364	**47**	1 67 2098	**67**	1 82 6075	**87**	1 93 9519
8	0 90 3090	**28**	1 44 7158	**48**	1 68 1241	**68**	1 83 2509	**88**	1 94 4483
9	0.95 4243	**29**	1.46 2398	**49**	1 69 0196	**69**	1.83 8849	**89**	1.94 9390
10	1.00 0000	**30**	1 47 7121	**50**	1 69 8970	**70**	1.84 5098	**90**	1 95 4243
11	1 04 1393	**31**	1 49 1362	**51**	1 70 7570	**71**	1.85 1258	**91**	1 95 9041
12	1.07 9181	**32**	1.50 5150	**52**	1 71 6003	**72**	1.85 7332	**92**	1 96 3788
13	1 11 3943	**33**	1 51 8514	**53**	1.72 4276	**73**	1 86 3323	**93**	1 96 8483
14	1.14 6128	**34**	1.53 1479	**54**	1.73 2394	**74**	1.86 9232	**94**	1 97 3128
15	1.17 6091	**35**	1.54 4068	**55**	1 74 0363	**75**	1.87 5061	**95**	1 97 7724
16	1.20 4120	**36**	1.55 6303	**56**	1 74 8188	**76**	1 88 0814	**96**	1 98 2271
17	1.23 0449	**37**	1.56 8202	**57**	1.75 5875	**77**	1 88 6491	**97**	1 98 6772
18	1.25 5273	**38**	1 57 9784	**58**	1 76 3428	**78**	1 89 2095	**98**	1 99 1226
19	1.27 8754	**39**	1.59 1065	**59**	1.77 0852	**79**	1.89 7627	**99**	1 99 5635

NOTE. Logarithms are given without interpolation for numbers between 10,000 and 10,999 on the first two pages, and for numbers between 1100 and 9999 on the remaining pages.

N.	0	1	2	3	4	5	6	7	8	9	Diff
1000	00 0000	0043	0087	0130	0174	0217	0260	0304	0347	0391	
01	0434	0477	0521	0564	0608	0651	0694	0738	0781	0824	
02	0868	0911	0954	0998	1041	1084	1128	1171	1214	1258	
03	1301	1344	1388	1431	1474	1517	1561	1604	1647	1690	
04	1734	1777	1820	1863	1907	1950	1993	2036	2080	2123	
05	00 2166	2209	2252	2296	2339	2382	2425	2468	2512	2555	
06	2598	2641	2684	2727	2771	2814	2857	2900	2943	2986	
07	3029	3073	3116	3159	3202	3245	3288	3331	3374	3417	
08	3461	3504	3547	3590	3633	3676	3719	3762	3805	3848	
09	3891	3934	3977	4020	4063	4106	4149	4192	4235	4278	
1010	00 4321	4364	4407	4450	4493	4536	4579	4622	4665	4708	44
11	4751	4794	4837	4880	4923	4966	5009	5052	5095	5138	
12	5181	5223	5266	5309	5352	5395	5438	5481	5524	5567	
13	5609	5652	5695	5738	5781	5824	5867	5909	5952	5995	
14	6038	6081	6124	6166	6209	6252	6295	6338	6380	6423	
1015	00 6466	6509	6552	6594	6637	6680	6723	6765	6808	6851	
16	6894	6936	6979	7022	7065	7107	7150	7193	7236	7278	
17	7321	7364	7406	7449	7492	7534	7577	7620	7662	7705	
18	7748	7790	7833	7876	7918	7961	8004	8046	8089	8132	
19	8174	8217	8259	8302	8345	8387	8430	8472	8515	8558	
1020	00 8600	8643	8685	8728	8770	8813	8856	8898	8941	8983	43
21	9026	9068	9111	9153	9196	9238	9281	9323	9366	9408	
22	9451	9493	9536	9578	9621	9663	9706	9748	9791	9833	
23	9876	9918	9961	*0003	*0045	*0088	*0130	*0173	*0215	*0258	
24	01 0300	0342	0385	0427	0470	0512	0554	0597	0639	0681	
1025	01 0724	0766	0809	0851	0893	0936	0978	1020	1063	1105	
26	1147	1190	1232	1274	1317	1359	1401	1444	1486	1528	
27	1570	1613	1655	1697	1740	1782	1824	1866	1909	1951	
28	1993	2035	2078	2120	2162	2204	2247	2289	2331	2373	
29	2415	2458	2500	2542	2584	2626	2669	2711	2753	2795	
1030	01 2837	2879	2922	2964	3006	3048	3090	3132	3174	3217	42
31	3259	3301	3343	3385	3427	3469	3511	3553	3596	3638	
32	3680	3722	3764	3806	3848	3890	3932	3974	4016	4058	
33	4100	4142	4184	4226	4268	4310	4353	4395	4437	4479	
34	4521	4563	4605	4647	4689	4730	4772	4814	4856	4898	
1035	01 4940	4982	5024	5066	5108	5150	5192	5234	5276	5318	
36	5360	5402	5444	5485	5527	5569	5611	5653	5695	5737	
37	5779	5821	5863	5904	5946	5988	6030	6072	6114	6156	
38	6197	6239	6281	6323	6365	6407	6448	6490	6532	6574	
39	6616	6657	6699	6741	6783	6824	6866	6908	6950	6992	
1040	01 7033	7075	7117	7159	7200	7242	7284	7326	7367	7409	41
41	7451	7492	7534	7576	7618	7659	7701	7743	7784	7826	
42	7868	7909	7951	7993	8034	8076	8118	8159	8201	8243	
43	8284	8326	8368	8409	8451	8492	8534	8576	8617	8659	
44	8700	8742	8784	8825	8867	8908	8950	8992	9033	9075	
1045	01 9116	9158	9199	9241	9282	9324	9366	9407	9449	9490	
46	9532	9573	9615	9656	9698	9739	9781	9822	9864	9905	
47	9947	9988	*0030	*0071	*0113	*0154	*0195	*0237	*0278	*0320	
48	02 0361	0403	0444	0486	0527	0568	0610	0651	0693	0734	
49	0775	0817	0858	0900	0941	0982	1024	1065	1107	1148	

PROPORTIONAL PARTS

Diff.	1	2	3	4	5	6	7	8	9
44	4.4	8.8	13.2	17.6	22.0	26.4	30.8	35.2	39.6
43	4.3	8.6	12.9	17.2	21.5	25.8	30.1	34.4	38.7
42	4.2	8.4	12.6	16.8	21.0	25.2	29.4	33.6	37.8
41	4.1	8.2	12.3	16.4	20.5	24.6	28.7	32.8	36.9

N.	0	1	2	3	4	5	6	7	8	9	Diff
1050	02 1189	1231	1272	1313	1355	1396	1437	1479	1520	1561	
51	1603	1644	1685	1727	1768	1809	1851	1892	1933	1974	
52	2016	2057	2098	2140	2182	2222	2263	2305	2346	2387	42
53	2428	2470	2511	2552	2593	2635	2676	2717	2758	2799	
54	2841	2882	2923	2964	3005	3047	3088	3129	3170	3211	
1055	02 3252	3294	3335	3376	3417	3458	3499	3541	3582	3623	
56	3664	3705	3746	3787	3828	3870	3911	3952	3993	4034	
57	4075	4116	4157	4198	4239	4280	4321	4363	4404	4445	
58	4486	4527	4568	4609	4650	4691	4732	4773	4814	4855	
59	4896	4937	4978	5019	5060	5101	5142	5183	5224	5265	
1060	02 5306	5347	5388	5429	5470	5511	5552	5593	5634	5674	
61	5715	5756	5797	5838	5879	5920	5961	6002	6043	6084	
62	6125	6165	6206	6247	6288	6329	6370	6411	6452	6492	41
63	6533	6574	6615	6656	6697	6737	6778	6819	6860	6901	
64	6942	6982	7023	7064	7105	7146	7186	7227	7268	7309	
1065	02 7350	7390	7431	7472	7513	7553	7594	7635	7676	7716	
66	7757	7798	7839	7879	7920	7961	8002	8042	8083	8124	
67	8164	8205	8246	8287	8327	8368	8409	8449	8490	8531	
68	8571	8612	8653	8693	8734	8775	8815	8856	8896	8937	
69	8978	9018	9059	9100	9140	9181	9221	9262	9303	9343	
1070	02 9384	9424	9465	9506	9546	9587	9627	9668	9708	9749	
71	9789	9830	9871	9911	9952	9992	*0033	*0073	*0114	*0154	
72	03 0195	0235	0276	0316	0357	0397	0438	0478	0519	0559	
73	0600	0640	0681	0721	0762	0802	0843	0883	0923	0964	
74	1004	1045	1085	1126	1166	1206	1247	1287	1328	1368	
1075	03 1408	1449	1489	1530	1570	1610	1651	1691	1732	1772	40
76	1812	1853	1893	1933	1974	2014	2054	2095	2135	2175	
77	2216	2256	2296	2337	2377	2417	2458	2498	2538	2578	
78	2619	2659	2699	2740	2780	2820	2860	2901	2941	2981	
79	3021	3062	3102	3142	3182	3223	3263	3303	3343	3384	
1080	03 3424	3464	3504	3544	3585	3625	3665	3705	3745	3786	
81	3826	3866	3906	3946	3986	4027	4067	4107	4147	4187	
82	4227	4267	4308	4348	4388	4428	4468	4508	4548	4588	
83	4628	4669	4709	4749	4789	4829	4869	4909	4949	4989	
84	5029	5069	5109	5149	5190	5230	5270	5310	5350	5390	
1085	03 5430	5470	5510	5550	5590	5630	5670	5710	5750	5790	
86	5830	5870	5910	5950	5990	6030	6070	6110	6150	6190	
87	6230	6269	6309	6349	6389	6429	6469	6509	6549	6589	
88	6629	6669	6709	6749	6789	6828	6868	6908	6948	6988	
89	7028	7068	7108	7148	7187	7227	7267	7307	7347	7387	
1090	03 7426	7466	7506	7546	7586	7626	7665	7705	7745	7785	39
91	7825	7865	7904	7944	7984	8024	8064	8103	8143	8183	
92	8223	8262	8302	8342	8382	8421	8461	8501	8541	8580	
93	8620	8660	8700	8739	8779	8819	8859	8898	8938	8978	
94	9017	9057	9097	9136	9176	9216	9255	9295	9335	9374	
1095	03 9414	9454	9493	9533	9573	9612	9652	9692	9731	9771	
96	9811	9850	9890	9929	9969	*0009	*0048	*0088	*0127	*0167	
97	04 0207	0246	0286	0325	0365	0405	0444	0484	0523	0563	
98	0602	0642	0681	0721	0761	0800	0840	0879	0919	0958	
99	0998	1037	1077	1116	1156	1195	1235	1274	1314	1353	

PROPORTIONAL PARTS

Diff.	1	2	3	4	5	6	7	8	9
42	4.2	8.4	12.6	16.8	21.0	25.2	29.4	33.6	37.8
41	4.1	8.2	12.3	16.4	20.5	24.6	28.7	32.8	36.9
40	4.0	8.0	12.0	16.0	20.0	24.0	28.0	32.0	36.0
39	3.9	7.8	11.7	15.6	19.5	23.4	27.3	31.2	35.1

N.	0	1	2	3	4	5	6	7	8	9	Diff
110	04 1393	1787	2182	2576	2969	3362	3755	4148	4540	4932	393
1	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830	390
2	9218	9606	9993	*0380	*0766	*1153	*1538	*1924	*2309	*2694	386
3	05 3078	3463	3846	4230	4613	4996	5378	5760	6142	6524	383
4	6905	7286	7666	8046	8426	8805	9185	9563	9942	*0320	379
115	06 0698	1075	1452	1829	2206	2582	2958	3333	3709	4083	376
6	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	373
7	8186	8557	8928	9298	9668	*0038	*0407	*0776	*1145	*1514	370
8	07 1882	2250	2617	2985	3352	3718	4085	4451	4816	5182	366
9	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	363

PROPORTIONAL PARTS

Diff.	1	2	3	4	5	6	7	8	9
395	39.5	79.0	118.5	158.0	197.5	237.0	276.5	316.0	355.5
394	39.4	78.8	118.2	157.6	197.0	236.4	275.8	315.2	354.6
393	39.3	78.6	117.9	157.2	196.5	235.8	275.1	314.4	353.7
392	39.2	78.4	117.6	156.8	196.0	235.2	274.4	313.6	352.8
391	39.1	78.2	117.3	156.4	195.5	234.6	273.7	312.8	351.9
390	39.0	78.0	117.0	156.0	195.0	234.0	273.0	312.0	351.0
389	38.9	77.8	116.7	155.6	194.5	233.4	272.3	311.2	350.1
388	38.8	77.6	116.4	155.2	194.0	232.8	271.6	310.4	349.2
387	38.7	77.4	116.1	154.8	193.5	232.2	270.9	309.6	348.3
386	38.6	77.2	115.8	154.4	193.0	231.6	270.2	308.8	347.4
385	38.5	77.0	115.5	154.0	192.5	231.0	269.5	308.0	346.5
384	38.4	76.8	115.2	153.6	192.0	230.4	268.8	307.2	345.6
383	38.3	76.6	114.9	153.2	191.5	229.8	268.1	306.4	344.7
382	38.2	76.4	114.6	152.8	191.0	229.2	267.4	305.6	343.8
381	38.1	76.2	114.3	152.4	190.5	228.6	266.7	304.8	342.9
380	38.0	76.0	114.0	152.0	190.0	228.0	266.0	304.0	342.0
379	37.9	75.8	113.7	151.6	189.5	227.4	265.3	303.2	341.1
378	37.8	75.6	113.4	151.2	189.0	226.8	264.6	302.4	340.2
377	37.7	75.4	113.1	150.8	188.5	226.2	263.9	301.6	339.3
376	37.6	75.2	112.8	150.4	188.0	225.6	263.2	300.8	338.4
375	37.5	75.0	112.5	150.0	187.5	225.0	262.5	300.0	337.5
374	37.4	74.8	112.2	149.6	187.0	224.4	261.8	299.2	336.6
373	37.3	74.6	111.9	149.2	186.5	223.8	261.1	298.4	335.7
372	37.2	74.4	111.6	148.8	186.0	223.2	260.4	297.6	334.8
371	37.1	74.2	111.3	148.4	185.5	222.6	259.7	296.8	333.9
370	37.0	74.0	111.0	148.0	185.0	222.0	259.0	296.0	333.0
369	36.9	73.8	110.7	147.6	184.5	221.4	258.3	295.2	332.1
368	36.8	73.6	110.4	147.2	184.0	220.8	257.6	294.4	331.2
367	36.7	73.4	110.1	146.8	183.5	220.2	256.9	293.6	330.3
366	36.6	73.2	109.8	146.4	183.0	219.6	256.2	292.8	329.4
365	36.5	73.0	109.5	146.0	182.5	219.0	255.5	292.0	328.5
364	36.4	72.8	109.2	145.6	182.0	218.4	254.8	291.2	327.6
363	36.3	72.6	108.9	145.2	181.5	217.8	254.1	290.4	326.7
362	36.2	72.4	108.6	144.8	181.0	217.2	253.4	289.6	325.8
361	36.1	72.2	108.3	144.4	180.5	216.6	252.7	288.8	324.9
360	36.0	72.0	108.0	144.0	180.0	216.0	252.0	288.0	324.0
359	35.9	71.8	107.7	143.6	179.5	215.4	251.3	287.2	323.1
358	35.8	71.6	107.4	143.2	179.0	214.8	250.6	286.4	322.2
357	35.7	71.4	107.1	142.8	178.5	214.2	249.9	285.6	321.3
356	35.6	71.2	106.8	142.4	178.0	213.6	249.2	284.8	320.4

N.	0	1	2	3	4	5	6	7	8	9	Diff.
120	07 9181	9543	9904	*0266	*0626	*0987	*1347	*1707	*2067	*2426	360
1	08 2785	3144	3503	3861	4219	4576	4934	5291	5647	6004	357
2	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552	355
3	9905	*0258	*0611	*0963	*1315	*1667	*2018	*2370	*2721	*3071	352
4	09 3422	3772	4122	4471	4820	5169	5518	5866	6215	6562	349
125	09 6910	7257	7604	7951	8298	8644	8990	9335	9681	*0026	346
6	10 0371	0715	1059	1403	1747	2091	2434	2777	3119	3462	343
7	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871	341
8	7210	7549	7888	8227	8565	8903	9241	9579	9916	*0253	338
9	11 0590	0926	1263	1599	1934	2270	2605	2940	3275	3609	335
130	11 3943	4277	4611	4944	5278	5611	5943	6276	6608	6940	333
1	7271	7603	7934	8265	8595	8926	9256	9586	9915	*0245	330
2	12 0574	0903	1231	1560	1888	2216	2544	2871	3198	3525	328
3	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781	325
4	7105	7429	7753	8076	8399	8722	9045	9368	9690	*0012	323

PROPORTIONAL PARTS

Diff.	1	2	3	4	5	6	7	8	9
355	35.5	71.0	106.5	142.0	177.5	213.0	248.5	284.0	319.5
354	35.4	70.8	106.2	141.6	177.0	212.4	247.8	283.2	318.6
353	35.3	70.6	105.9	141.2	176.5	211.8	247.1	282.4	317.7
352	35.2	70.4	105.6	140.8	176.0	211.2	246.4	281.6	316.8
351	35.1	70.2	105.3	140.4	175.5	210.6	245.7	280.8	315.9
350	35.0	70.0	105.0	140.0	175.0	210.0	245.0	280.0	315.0
349	34.9	69.8	104.7	139.6	174.5	209.4	244.3	279.2	314.1
348	34.8	69.6	104.4	139.2	174.0	208.8	243.6	278.4	313.2
347	34.7	69.4	104.1	138.8	173.5	208.2	242.9	277.6	312.3
346	34.6	69.2	103.8	138.4	173.0	207.6	242.2	276.8	311.4
345	34.5	69.0	103.5	138.0	172.5	207.0	241.5	276.0	310.5
344	34.4	68.8	103.2	137.6	172.0	206.4	240.8	275.2	309.6
343	34.3	68.6	102.9	137.2	171.5	205.8	240.1	274.4	308.7
342	34.2	68.4	102.6	136.8	171.0	205.2	239.4	273.6	307.8
341	34.1	68.2	102.3	136.4	170.5	204.6	238.7	272.8	306.9
340	34.0	68.0	102.0	136.0	170.0	204.0	238.0	272.0	306.0
339	33.9	67.8	101.7	135.6	169.5	203.4	237.3	271.2	305.1
338	33.8	67.6	101.4	135.2	169.0	202.8	236.6	270.4	304.2
337	33.7	67.4	101.1	134.8	168.5	202.2	235.9	269.6	303.3
336	33.6	67.2	100.8	134.4	168.0	201.6	235.2	268.8	302.4
335	33.5	67.0	100.5	134.0	167.5	201.0	234.5	268.0	301.5
334	33.4	66.8	100.2	133.6	167.0	200.4	233.8	267.2	300.6
333	33.3	66.6	99.9	133.2	166.5	199.8	233.1	266.4	299.7
332	33.2	66.4	99.6	132.8	166.0	199.2	232.4	265.6	298.8
331	33.1	66.2	99.3	132.4	165.5	198.6	231.7	264.8	297.9
330	33.0	66.0	99.0	132.0	165.0	198.0	231.0	264.0	297.0
329	32.9	65.8	98.7	131.6	164.5	197.4	230.3	263.2	296.1
328	32.8	65.6	98.4	131.2	164.0	196.8	229.6	262.4	295.2
327	32.7	65.4	98.1	130.8	163.5	196.2	228.9	261.6	294.3
326	32.6	65.2	97.8	130.4	163.0	195.6	228.2	260.8	293.4
325	32.5	65.0	97.5	130.0	162.5	195.0	227.5	260.0	292.5
324	32.4	64.8	97.2	129.6	162.0	194.4	226.8	259.2	291.6
323	32.3	64.6	96.9	129.2	161.5	193.8	226.1	258.4	290.7
322	32.2	64.4	96.6	128.8	161.0	193.2	225.4	257.6	289.8

N.	0	1	2	3	4	5	6	7	8	9	Diff
135	13 0334	0655	0977	1298	1619	1939	2260	2580	2900	3219	321
6	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403	318
7	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564	316
8	9879	*0194	*0508	*0822	*1136	*1450	*1763	*2076	*2389	*2702	314
9	14 3015	3327	3639	3951	4263	4574	4885	5196	5507	5818	311
140	14 6128	6438	6748	7058	7367	7676	7985	8294	8603	8911	309
1	9219	9527	9835	*0142	*0449	*0756	*1063	*1370	*1676	*1982	307
2	15 2288	2594	2900	3205	3510	3815	4120	4424	4728	5032	305
3	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061	303
4	8362	8664	8965	9266	9567	9868	*0168	*0469	*0769	*1068	301
145	16 1368	1667	1967	2266	2564	2863	3161	3460	3758	4055	299
6	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022	297
7	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968	295
8	17 0262	0555	0848	1141	1434	1726	2019	2311	2603	2895	293
9	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802	291

PROPORTIONAL PARTS

Diff.	1	2	3	4	5	6	7	8	9
322	32.2	64.4	96.6	128.8	161.0	193.2	225.4	257.6	289.8
321	32.1	64.2	96.3	128.4	160.5	192.6	224.7	256.8	288.9
320	32.0	64.0	96.0	128.0	160.0	192.0	224.0	256.0	288.0
319	31.9	63.8	95.7	127.6	159.5	191.4	223.3	255.2	287.1
318	31.8	63.6	95.4	127.2	159.0	190.8	222.6	254.4	286.2
317	31.7	63.4	95.1	126.8	158.5	190.2	221.9	253.6	285.3
316	31.6	63.2	94.8	126.4	158.0	189.6	221.2	252.8	284.4
315	31.5	63.0	94.5	126.0	157.5	189.0	220.5	252.0	283.5
314	31.4	62.8	94.2	125.6	157.0	188.4	219.8	251.2	282.6
313	31.3	62.6	93.9	125.2	156.5	187.8	219.1	250.4	281.7
312	31.2	62.4	93.6	124.8	156.0	187.2	218.4	249.6	280.8
311	31.1	62.2	93.3	124.4	155.5	186.6	217.7	248.8	279.9
310	31.0	62.0	93.0	124.0	155.0	186.0	217.0	248.0	279.0
309	30.9	61.8	92.7	123.6	154.5	185.4	216.3	247.2	278.1
308	30.8	61.6	92.4	123.2	154.0	184.8	215.6	246.4	277.2
307	30.7	61.4	92.1	122.8	153.5	184.2	214.9	245.6	276.3
306	30.6	61.2	91.8	122.4	153.0	183.6	214.2	244.8	275.4
305	30.5	61.0	91.5	122.0	152.5	183.0	213.5	244.0	274.5
304	30.4	60.8	91.2	121.6	152.0	182.4	212.8	243.2	273.6
303	30.3	60.6	90.9	121.2	151.5	181.8	212.1	242.4	272.7
302	30.2	60.4	90.6	120.8	151.0	181.2	211.4	241.6	271.8
301	30.1	60.2	90.3	120.4	150.5	180.6	210.7	240.8	270.9
300	30.0	60.0	90.0	120.0	150.0	180.0	210.0	240.0	270.0
299	29.9	59.8	89.7	119.6	149.5	179.4	209.3	239.2	269.1
298	29.8	59.6	89.4	119.2	149.0	178.8	208.6	238.4	268.2
297	29.7	59.4	89.1	118.8	148.5	178.2	207.9	237.6	267.3
296	29.6	59.2	88.8	118.4	148.0	177.6	207.2	236.8	266.4
295	29.5	59.0	88.5	118.0	147.5	177.0	206.5	236.0	265.5
294	29.4	58.8	88.2	117.6	147.0	176.4	205.8	235.2	264.6
293	29.3	58.6	87.9	117.2	146.5	175.8	205.1	234.4	263.7
292	29.2	58.4	87.6	116.8	146.0	175.2	204.4	233.6	262.8
291	29.1	58.2	87.3	116.4	145.5	174.6	203.7	232.8	261.9
290	29.0	58.0	87.0	116.0	145.0	174.0	203.0	232.0	261.0
289	28.9	57.8	86.7	115.6	144.5	173.4	202.3	231.2	260.1
288	28.8	57.6	86.4	115.2	144.0	172.8	201.6	230.4	259.2
287	28.7	57.4	86.1	114.8	143.5	172.2	200.9	229.6	258.3
286	28.6	57.2	85.8	114.4	143.0	171.6	200.2	228.8	257.4

Table XV—Six-Place Logarithms

N.	0	1	2	3	4	5	6	7	8	9	Diff.
150	17 6091	6381	6670	6959	7248	7536	7825	8113	8401	8689	289
1	8977	9264	9552	9839	*0126	*0413	*0699	*0986	*1272	*1558	287
2	18 1844	2129	2415	2700	2985	3270	3555	3839	4123	4407	285
3	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239	283
4	7521	7803	8084	8366	8647	8928	9209	9490	9771	*0051	281
155	19 0332	0612	0892	1171	1451	1730	2010	2289	2567	2846	279
6	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623	278
7	5900	6176	6453	6729	7005	7281	7556	7832	8107	8382	276
8	8657	8932	9206	9481	9755	*0029	*0303	*0577	*0850	*1124	274
9	20 1397	1670	1943	2216	2488	2761	3033	3305	3577	3848	272
160	20 4120	4391	4663	4934	5204	5475	5746	6016	6286	6556	271
1	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247	269
2	9515	9783	*0051	*0319	*0586	*0853	*1121	*1388	*1654	*1921	267
3	21 2188	2454	2720	2986	3252	3518	3783	4049	4314	4579	266
4	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221	264
165	21 7484	7747	8010	8273	8536	8798	9060	9323	9585	9846	262
6	22 0108	0370	0631	0892	1153	1414	1675	1936	2196	2456	261
7	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051	259
8	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630	258
9	7887	8144	8400	8657	8913	9170	9426	9682	9938	*0193	256

PROPORTIONAL PARTS

Diff.	1	2	3	4	5	6	7	8	9
285	28.5	57.0	85.5	114.0	142.5	171.0	199.5	228.0	256.5
284	28.4	56.8	85.2	113.6	142.0	170.4	198.8	227.2	255.6
283	28.3	56.6	84.9	113.2	141.5	169.8	198.1	226.4	254.7
282	28.2	56.4	84.6	112.8	141.0	169.2	197.4	225.6	253.8
281	28.1	56.2	84.3	112.4	140.5	168.6	196.7	224.8	252.9
280	28.0	56.0	84.0	112.0	140.0	168.0	196.0	224.0	252.0
279	27.9	55.8	83.7	111.6	139.5	167.4	195.3	223.2	251.1
278	27.8	55.6	83.4	111.2	139.0	166.8	194.6	222.4	250.2
277	27.7	55.4	83.1	110.8	138.5	166.2	193.9	221.6	249.3
276	27.6	55.2	82.8	110.4	138.0	165.6	193.2	220.8	248.4
275	27.5	55.0	82.5	110.0	137.5	165.0	192.5	220.0	247.5
274	27.4	54.8	82.2	109.6	137.0	164.4	191.8	219.2	246.6
273	27.3	54.6	81.9	109.2	136.5	163.8	191.1	218.4	245.7
272	27.2	54.4	81.6	108.8	136.0	163.2	190.4	217.6	244.8
271	27.1	54.2	81.3	108.4	135.5	162.6	189.7	216.8	243.9
270	27.0	54.0	81.0	108.0	135.0	162.0	189.0	216.0	243.0
269	26.9	53.8	80.7	107.6	134.5	161.4	188.3	215.2	242.1
268	26.8	53.6	80.4	107.2	134.0	160.8	187.6	214.4	241.2
267	26.7	53.4	80.1	106.8	133.5	160.2	186.9	213.6	240.3
266	26.6	53.2	79.8	106.4	133.0	159.6	186.2	212.8	239.4
265	26.5	53.0	79.5	106.0	132.5	159.0	185.5	212.0	238.5
264	26.4	52.8	79.2	105.6	132.0	158.4	184.8	211.2	237.6
263	26.3	52.6	78.9	105.2	131.5	157.8	184.1	210.4	236.7
262	26.2	52.4	78.6	104.8	131.0	157.2	183.4	209.6	235.8
261	26.1	52.2	78.3	104.4	130.5	156.6	182.7	208.8	234.9
260	26.0	52.0	78.0	104.0	130.0	156.0	182.0	208.0	234.0
259	25.9	51.8	77.7	103.6	129.5	155.4	181.3	207.2	233.1
258	25.8	51.6	77.4	103.2	129.0	154.8	180.6	206.4	232.2
257	25.7	51.4	77.1	102.8	128.5	154.2	179.9	205.6	231.3
256	25.6	51.2	76.8	102.4	128.0	153.6	179.2	204.8	230.4
255	25.5	51.0	76.5	102.0	127.5	153.0	178.5	204.0	229.5

N.	0	1	2	3	4	5	6	7	8	9	Diff
170	23 0449	0704	0960	1215	1470	1724	1979	2234	2488	2742	255
1	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276	253
2	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795	252
3	8046	8297	8548	8799	9049	9299	9550	9800	*0050	*0300	250
4	24 0549	0799	1048	1297	1546	1795	2044	2293	2541	2790	249
175	24 3038	3286	3534	3782	4030	4277	4525	4772	5019	5266	248
6	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728	246
7	7973	8219	8464	8709	8954	9198	9443	9687	9932	*0176	245
8	25 0420	0664	0908	1151	1395	1638	1881	2125	2368	2610	243
9	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031	242
180	25 5273	5514	5755	5996	6237	6477	6718	6958	7198	7439	241
1	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833	239
2	26 0071	0310	0548	0787	1025	1263	1501	1739	1976	2214	238
3	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582	237
4	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937	235
185	26 7172	7406	7641	7875	8110	8344	8578	8812	9046	9279	234
6	9513	9746	9980	*0213	*0446	*0679	*0912	*1144	*1377	*1609	233
7	27 1842	2074	2306	2538	2770	3001	3233	3464	3696	3927	232
8	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232	230
9	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525	229
190	27 8754	8982	9211	9439	9667	9895	*0123	*0351	*0578	*0806	228
1	28 1033	1261	1488	1715	1942	2169	2396	2622	2849	3075	227
2	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332	226
3	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578	225
4	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812	223

PROPORTIONAL PARTS

Diff.	1	2	3	4	5	6	7	8	9
255	25.5	51.0	76.5	102.0	127.5	153.0	178.5	204.0	229.5
254	25.4	50.8	76.2	101.6	127.0	152.4	177.8	203.2	228.6
253	25.3	50.6	75 9	101.2	126.5	151.8	177.1	202.4	227.7
252	25.2	50.4	75.6	100.8	126.0	151.2	176.4	201.6	226.8
251	25.1	50.2	75.3	100.4	125.5	150.6	175.7	200.8	225.9
250	25.0	50.0	75.0	100.0	125.0	150.0	175.0	200.0	225.0
249	24.9	49.8	74.7	99.6	124 5	149.4	174.3	199.2	224.1
248	24.8	49.6	74.4	99.2	124 0	148.8	173.6	198.4	223.2
247	24.7	49.4	74.1	98.8	123.5	148.2	172.9	197.6	222.3
246	24.6	49.2	73.8	98.4	123.0	147.6	172.2	196.8	221.4
245	24.5	49.0	73.5	98.0	122.5	147.0	171.5	196.0	220.5
244	24.4	48.8	73.2	97.6	122.0	146.4	170.8	195.2	219.6
243	24.3	48.6	72.9	97.2	121.5	145.8	170.1	194.4	218.7
242	24.2	48.4	72.6	96.8	121 0	145.2	169.4	193.6	217.8
241	24.1	48.2	72.3	96.4	120.5	144.6	168.7	192.8	216.9
240	24.0	48.0	72.0	96.0	120.0	144.0	168.0	192.0	216.0
239	23.9	47.8	71.7	95.6	119.5	143.4	167.3	191.2	215.1
238	23.8	47.6	71.4	95.2	119.0	142.8	166.6	190.4	214.2
237	23.7	47.4	71.1	94.8	118.5	142.2	165.9	189.6	213.3
236	23.6	47.2	70.8	94.4	118.0	141.6	165.2	188.8	212.4
235	23.5	47.0	70.5	94.0	117.5	141.0	164.5	188.0	211.5
234	23.4	46.8	70.2	93.6	117.0	140.4	163.8	187.2	210.6
233	23.3	46.6	69.9	93.2	116 5	139.8	163.1	186.4	209 7
232	23.2	46.4	69.6	92.8	116.0	139.2	162.4	185.6	208.8
231	23.1	46.2	69.3	92.4	115.5	138.6	161.7	184.8	207.9
230	23.0	46.0	69.0	92.0	115.0	138.0	161.0	184.0	207.0
229	22.9	45.8	67.8	91.6	114.5	137.4	160.3	183.2	206.1
228	22.8	45.6	68.4	91.2	114.0	136.8	159.6	182.4	205.2
227	22.7	45.4	68.1	90.8	113.5	136.2	158.9	181.6	204.3

Table XV—Six-Place Logarithms

N.	0	1	2	3	4	5	6	7	8	9	Diff
195	29 0035	0257	0480	0702	0925	1147	1369	1591	1813	2034	222
6	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246	221
7	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446	220
8	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635	219
9	8853	9071	9289	9507	9725	9943	*0161	*0378	*0595	*0813	218
200	30 1030	1247	1464	1681	1898	2114	2331	2547	2764	2980	217
1	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136	216
2	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282	215
3	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417	213
4	9630	9843	*0056	*0268	*0481	*0693	*0906	*1118	*1330	*1542	212
205	31 1754	1966	2177	2389	2600	2812	3023	3234	3445	3656	211
6	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760	210
7	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854	209
8	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938	208
9	32 0146	0354	0562	0769	0977	1184	1391	1598	1805	2012	207
210	32 2219	2426	2633	2839	3046	3252	3458	3665	3871	4077	206
1	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131	205
2	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176	204
3	8380	8583	8787	8991	9194	9398	9601	9805	*0008	*0211	203
4	33 0414	0617	0819	1022	1225	1427	1630	1832	2034	2236	202
215	33 2438	2640	2842	3044	3246	3447	3649	3850	4051	4253	202
6	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260	201
7	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257	200
8	8456	8656	8855	9054	9253	9451	9650	9849	*0047	*0246	199
9	34 0444	0642	0841	1039	1237	1435	1632	1830	2028	2225	198

PROPORTIONAL PARTS

Diff.	1	2	3	4	5	6	7	8	9
226	22.6	45.2	67.8	90.4	113.0	135.6	158.2	180.8	203 4
225	22.5	45.0	67.5	90.0	112.5	135.0	157.5	180.0	202.5
224	22.4	44.8	67.2	89.6	112.0	134.4	156 8	179.2	201.6
223	22.3	44.6	66.9	89.2	111.5	133.8	156.1	178.4	200.7
222	22.2	44.4	66.6	88.8	111.0	133.2	155.4	177.6	199.8
221	22.1	44.2	66.3	88.4	110.5	132.6	154 7	176.8	198.9
220	22.0	44.0	66.0	88.0	110.0	132.0	154.0	176.0	198 0
219	21.9	43.8	65.7	87.6	109.5	131.4	153.3	175.2	197.1
218	21.8	43 6	65.4	87.2	109.0	130.8	152.6	174.4	196.2
217	21.7	43.4	65.1	86.8	108.5	130.2	151.9	173.6	195.3
216	21.6	43.2	64.8	86.4	108.0	129.6	151.2	172.8	194.4
215	21.5	43.0	64.5	86.0	107.5	129.0	150 5	172.0	193.5
214	21.4	42.8	64.2	85.6	107.0	128.4	149.8	171.2	192.6
213	21.3	42.6	63.9	85.2	106.5	127.8	149.1	170.4	191.7
212	21.2	42.4	63.6	84.8	106.0	127.2	148.4	169.6	190.8
211	21.1	42.2	63.3	84.4	105.5	126.6	147.7	168.8	189.9
210	21.0	42.0	63.0	84.0	105.0	126.0	147.0	168.0	189.0
209	20.9	41.8	62.7	83.6	104.5	125.4	146.3	167.2	188.1
208	20.8	41.6	62.4	83.2	104.0	124.8	145.6	166.4	187.2
207	20.7	41.4	62.1	82.8	103.5	124.2	144.9	165.6	186.3
206	20.6	41.2	61.8	82.4	103.0	123.6	144 2	164.8	185.4
205	20.5	41.0	61.5	82.0	102.5	123.0	143 5	164.0	184.5
204	20.4	40.8	61.2	81.6	102.0	122.4	142.8	163.2	183.6
203	20.3	40.6	60.9	81.2	101.5	121.8	142 1	162.4	182.7
202	20.2	40.4	60.6	80.8	101.0	121.2	141.4	161.6	181.8
201	20.1	40.2	60.3	80.4	100.5	120.6	140.7	160.8	180.9
200	20.0	40.0	60.0	80.0	100.0	120.0	140.0	160.0	180.0
199	19.9	39.8	59.7	79.6	99.5	119.4	139.3	159.2	179.1
198	19.8	39 6	59.4	79.2	99.0	118.8	138.6	158.4	178.2

N.	0	1	2	3	4	5	6	7	8	9	Diff.
220	34 2423	2620	2817	3014	3212	3409	3606	3802	3999	4196	197
1	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157	196
2	6353	6549	6744	6939	7135	7330	7525	7720	7915	8110	195
3	8305	8500	8694	8889	9083	9278	9472	9666	9860	*0054	194
4	35 0248	0442	0636	0829	1023	1216	1410	1603	1796	1989	193
225	35 2183	2375	2568	2761	2954	3147	3339	3532	3724	3916	193
6	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834	192
7	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744	191
8	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646	190
9	9835	*0025	*0215	*0404	*0593	*0783	*0972	*1161	*1350	*1539	189
230	36 1728	1917	2105	2294	2482	2671	2859	3048	3236	3424	188
1	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301	188
2	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169	187
3	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030	186
4	9216	9401	9587	9772	9958	*0143	*0328	*0513	*0698	*0883	185
235	37 1068	1253	1437	1622	1806	1991	2175	2360	2544	2728	184
6	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565	184
7	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394	183
8	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216	182
9	8398	8580	8761	8943	9124	9306	9487	9668	9849	*0030	181
240	38 0211	0392	0573	0754	0934	1115	1296	1476	1656	1837	181
1	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636	180
2	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428	179
3	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212	178
4	7390	7568	7746	7923	8101	8279	8456	8634	8811	8989	178
245	38 9166	9343	9520	9698	9875	*0051	*0228	*0405	*0582	*0759	177
6	39 0935	1112	1288	1464	1641	1817	1993	2169	2345	2521	176
7	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277	176
8	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025	175
9	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766	174

PROPORTIONAL PARTS

Diff.	1	2	3	4	5	6	7	8	9
198	19.8	39.6	59.4	79.2	99.0	118.8	138.6	158.6	178.2
197	19.7	39.4	59.1	78.8	98.5	118.2	137.9	157.6	177.3
196	19.6	39.2	58.8	78.4	98.0	117.6	137.2	156.8	176.4
195	19.5	39.0	58.5	78.0	97.5	117.0	136.5	156.0	175.5
194	19.4	38.8	58.2	77.6	97.0	116.4	135.8	155.2	174.6
193	19.3	38 6	57.9	77.2	96.5	115.8	135.1	154.4	173.7
192	19.2	38.4	57.6	76.8	96.0	115.2	134.4	153.6	172.8
191	19.1	38.2	57.3	76.4	95.5	114.6	133.7	152.8	171.9
190	19.0	38.0	57.0	76.0	95.0	114.0	133.0	152.0	171.0
189	18.9	37.8	56 7	75.6	94.5	113.4	132.3	151.2	170.1
188	18.8	37.6	56.4	75.2	94.0	112.8	131.6	150.4	169.2
187	18.7	37.4	56.1	74.8	93.5	112.2	130.9	149.6	168.3
186	18.6	37.2	55.8	74.4	93.0	111.6	130.2	148.8	167.4
185	18.5	37.0	55.5	74.0	92.5	111.0	129.5	148.C	166.5
184	18.4	36.8	55 2	73.6	92.0	110.4	128.8	147.2	165.6
183	18.3	36.6	54 9	73.2	91.5	109.8	128.1	146.4	164.7
182	18.2	36.4	54.6	72.8	91.0	109.2	127.4	145.6	163.8
181	18.1	36.2	54.3	72.4	90.5	108.6	126.7	144.8	162.9
180	18.0	36.0	54.0	72.0	90.0	108.0	126.0	144.0	162.0
179	17.9	35.8	53.7	71.6	89.5	107.4	125.3	143.2	161.1
178	17.8	35.6	53.4	71.2	89.0	106 8	124.6	142.4	160.2
177	17.7	35.4	53.1	70.8	88.5	106.2	123.9	141.6	159.3
176	17.6	35.2	52.8	70.4	88.0	105.6	123.2	140.8	158.4
175	17.5	35.0	52.5	70.0	87.5	105.0	122.5	140.0	157.5

Table XV—Six-Place Logarithms

N.	0	1	2	3	4	5	6	7	8	9	Diff.
250	39 7940	8114	8287	8461	8634	8808	8981	9154	9328	9501	173
1	9674	9847	*0020	*0192	*0365	*0538	*0711	*0883	*1056	*1228	173
2	40 1401	1573	1745	1917	2089	2261	2433	2605	2777	2949	172
3	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663	171
4	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370	171
255	40 6540	6710	6881	7051	7221	7391	7561	7731	7901	8070	170
6	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764	169
7	9933	*0102	*0271	*0440	*0609	*0777	*0946	*1114	*1283	*1451	169
8	41 1620	1788	1956	2124	2293	2461	2629	2796	2964	3132	168
9	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806	167
260	41 4973	5140	5307	5474	5641	5808	5974	6141	6308	6474	167
1	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135	166
2	8301	8467	8633	8798	8964	9129	9295	9460	9625	9791	165
3	9956	*0121	*0286	*0451	*0616	*0781	*0945	*1110	*1275	*1439	165
4	42 1604	1768	1933	2097	2261	2426	2590	2754	2918	3082	164
265	42 3246	3410	3574	3737	3901	4065	4228	4392	4555	4718	164
6	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349	163
7	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973	162
8	8135	8297	8459	8621	8783	8944	9106	9268	9429	9591	162
9	9752	9914	*0075	*0236	*0398	*0559	*0720	*0881	*1042	*1203	161
270	43 1364	1525	1685	1846	2007	2167	2328	2488	2649	2809	161
1	2969	3130	3290	3450	3610	3770	3930	4090	4249	4409	160
2	4569	4729	4888	5048	5207	5367	5526	5685	5844	6004	159
3	6163	6322	6481	6640	6799	6957	7116	7275	7433	7592	159
4	7751	7909	8067	8226	8384	8542	8701	8859	9017	9175	158
275	43 9333	9491	9648	9806	9964	*0122	*0279	*0437	*0594	*0752	158
6	44 0909	1066	1224	1381	1538	1695	1852	2009	2166	2323	157
7	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889	157
8	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449	156
9	5604	5760	5915	6071	6226	6382	6537	6692	6848	7003	155

PROPORTIONAL PARTS

Diff.	1	2	3	4	5	6	7	8	9
174	17.4	34.8	52.2	69.6	87.0	104.4	121.8	139.2	156.6
173	17.3	34.6	51.9	69.2	86.5	103.8	121.1	138.4	155.7
172	17.2	34.4	51.6	68.8	86.0	103.2	120.4	137.6	154.8
171	17.1	34.2	51.3	68.4	85.5	102.6	119.7	136.8	153.9
170	17.0	34.0	51.0	68.0	85.0	102.0	119.0	136.0	153.0
169	16.9	33.8	50.7	67.6	84.5	101.4	118.3	135.2	152.1
168	16.8	33.6	50.4	67.2	84.0	100.8	117.6	134.4	151.2
167	16.7	33.4	50.1	66.8	83.5	100.2	116.9	133.6	150.3
166	16.6	33.2	49.8	66.4	83.0	99.6	116.2	132.8	149.4
165	16.5	33.0	49.5	66.0	82.5	99.0	115.5	132.0	148.5
164	16.4	32.8	49.2	65.6	82.0	98.4	114.8	131.2	147.6
163	16.3	32.6	48.9	65.2	81.5	97.8	114.1	130.4	146.7
162	16.2	32.4	48.6	64.8	81.0	97.2	113.4	129.6	145.8
161	16.1	32.2	48.3	64.4	80.5	96.6	112.7	128.8	144.9
160	16.0	32.0	48.0	64.0	80.0	96.0	112.0	128.0	144.0
159	15.9	31.8	47.7	63.6	79.5	95.4	111.3	127.2	143.1
158	15.8	31.6	47.4	63.2	79.0	94.8	110.6	126.4	142.2
157	15.7	31.4	47.1	62.8	78.5	94.2	109.9	125.6	141.3
156	15.6	31.2	46.8	62.4	78.0	93.6	109.2	124.8	140.4
155	15.5	31.0	46.5	62.0	77.5	93.0	108.5	124.0	139.5

N.	0	1	2	3	4	5	6	7	8	9	Diff.
280	44 7158	7313	7468	7623	7778	7933	8088	8242	8397	8552	155
1	8706	8861	9015	9170	9324	9478	9633	9787	9941	*0095	154
2	45 0249	0403	0557	0711	0865	1018	1172	1326	1479	1633	154
3	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165	153
4	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692	153
285	45 4845	4997	5150	5302	5454	5606	5758	5910	6062	6214	152
6	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731	152
7	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242	151
8	9392	9543	9694	9845	9995	*0146	*0296	*0447	*0597	*0748	151
9	46 0898	1048	1198	1348	1499	1649	1799	1948	2098	2248	150
290	46 2398	2548	2697	2847	2997	3146	3296	3445	3594	3744	150
1	3893	4042	4191	4340	4490	4639	4788	4936	5085	5234	149
2	5383	5532	5680	5829	5977	6126	6274	6423	6571	6719	149
3	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200	148
4	8347	8495	8643	8790	8938	9085	9233	9380	9527	9675	148
295	46 9822	9969	*0116	*0263	*0410	*0557	*0704	*0851	*0998	*1145	147
6	47 1292	1438	1585	1732	1878	2025	2171	2318	2464	2610	146
7	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071	146
8	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526	146
9	5671	5816	5962	6107	6252	6397	6542	6687	6832	6976	145
300	47 7121	7266	7411	7555	7700	7844	7989	8133	8278	8422	145
1	8566	8711	8855	8999	9143	9287	9431	9575	9719	9863	144
2	48 0007	0151	0294	0438	0582	0725	0869	1012	1156	1299	144
3	1443	1586	1729	1872	2016	2159	2302	2445	2588	2731	143
4	2874	3016	3159	3302	3445	3587	3730	3872	4015	4157	143
305	48 4300	4442	4585	4727	4869	5011	5153	5295	5437	5579	142
6	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997	142
7	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410	141
8	8551	8692	8833	8974	9114	9255	9396	9537	9677	9818	141
9	9958	*0099	*0239	*0380	*0520	*0661	*0801	*0941	*1081	*1222	140
310	49 1362	1502	1642	1782	1922	2062	2201	2341	2481	2621	140
1	2760	2900	3040	3179	3319	3458	3597	3737	3876	4015	139
2	4155	4294	4433	4572	4711	4850	4989	5128	5267	5406	139
3	5544	5683	5822	5960	6099	6238	6376	6515	6653	6791	139
4	6930	7068	7206	7344	7483	7621	7759	7897	8035	8173	138

PROPORTIONAL PARTS

Diff.	1	2	3	4	5	6	7	8	9
155	15.5	31.0	46.5	62.0	77.5	93.0	108.5	124.0	139.5
154	15.4	30.8	46.2	61.6	77.0	92.4	107.8	123.2	138.6
153	15.3	30.6	45.9	61.2	76.5	91.8	107.1	122.4	137.7
152	15.2	30.4	45.6	60.8	76.0	91.2	106.4	121.6	136.8
151	15.1	30.2	45.3	60.4	75.5	90.6	105.7	120.8	135.9
150	15.0	30.0	45.0	60.0	75.0	90.0	105.0	120.0	135.0
149	14.9	29.8	44.7	59.6	74.5	89.4	104.3	119.2	134.1
148	14.8	29.6	44.4	59.2	74.0	88.8	103.6	118.4	133.2
147	14.7	29.4	44.1	58.8	73.5	88.2	102.9	117.6	132.3
146	14.6	29.2	43.8	58.4	73.0	87.6	102.2	116.8	131.4
145	14.5	29.0	43.5	58.0	72.5	87.0	101.5	116.0	130.5
144	14.4	28.8	43.2	57.6	72.0	86.4	100.8	115.2	129.6
143	14.3	28.6	42.9	57.2	71.5	85.8	100.1	114.4	128.7
142	14.2	28.4	42.6	56.8	71.0	85.2	99.4	113.6	127.8
141	14.1	28.2	42.3	56.4	70.5	84.6	98.7	112.8	126.9
140	14.0	28.0	42.0	56.0	70.0	84.0	98.0	112.0	126.0
139	13.9	27.8	41.7	55.6	69.5	83.4	97.3	111.2	125.1

N.	0	1	2	3	4	5	6	7	8	9	Diff.
315	49 8311	8448	8586	8724	8862	8999	9137	9275	9412	9550	138
6	9687	9824	9962	*0099	*0236	*0374	*0511	*0648	*0785	*0922	137
7	50 1059	1196	1333	1470	1607	1744	1880	2017	2154	2291	137
8	2427	2564	2700	2837	2973	3109	3246	3382	3518	3655	136
9	3791	3927	4063	4199	4335	4471	4607	4743	4878	5014	136
320	50 5150	5286	5421	5557	5693	5828	5964	6099	6234	6370	136
1	6505	6640	6776	6911	7046	7181	7316	7451	7586	7721	135
2	7856	7991	8126	8260	8395	8530	8664	8799	8934	9068	135
3	9203	9337	9471	9606	9740	9874	*0009	*0143	*0277	*0411	134
4	51 0545	0679	0813	0947	1081	1215	1349	1482	1616	1750	134
325	51 1883	2017	2151	2284	2418	2551	2684	2818	2951	3084	133
6	3218	3351	3484	3617	3750	3883	4016	4149	4282	4415	133
7	4548	4681	4813	4946	5079	5211	5344	5476	5609	5741	133
8	5874	6006	6139	6271	6403	6535	6668	6800	6932	7064	132
9	7196	7328	7460	7592	7724	7855	7987	8119	8251	8382	132
330	51 8514	8646	8777	8909	9040	9171	9303	9434	9566	9697	131
1	9828	9959	*0090	*0221	*0353	*0484	*0615	*0745	*0876	*1007	131
2	52 1138	1269	1400	1530	1661	1792	1922	2053	2183	2314	131
3	2444	2575	2705	2835	2966	3096	3226	3356	3486	3616	130
4	3746	3876	4006	4136	4266	4396	4526	4656	4785	4915	130
335	52 5045	5174	5304	5434	5563	5693	5822	5951	6081	6210	129
6	6339	6469	6598	6727	6856	6985	7114	7243	7372	7501	129
7	7630	7759	7888	8016	8145	8274	8402	8531	8660	8788	129
8	8917	9045	9174	9302	9430	9559	9687	9815	9943	*0072	128
9	53 0200	0328	0456	0584	0712	0840	0968	1096	1223	1351	128
340	53 1479	1607	1734	1862	1990	2117	2245	2372	2500	2627	128
1	2754	2882	3009	3136	3264	3391	3518	3645	3772	3899	127
2	4026	4153	4280	4407	4534	4661	4787	4914	5041	5167	127
3	5294	5421	5547	5674	5800	5927	6053	6180	6306	6432	126
4	6558	6685	6811	6937	7063	7189	7315	7441	7567	7693	126
345	53 7819	7945	8071	8197	8322	8448	8574	8699	8825	8951	126
6	9076	9202	9327	9452	9578	9703	9829	9954	*0079	*0204	125
7	54 0329	0455	0580	0705	0830	0955	1080	1205	1330	1454	125
8	1579	1704	1829	1953	2078	2203	2327	2452	2576	2701	125
9	2825	2950	3074	3199	3323	3447	3571	3696	3820	3944	124

PROPORTIONAL PARTS

Diff.	1	2	3	4	5	6	7	8	9
138	13.8	27.6	41.4	55.2	69.0	82.8	96.6	110.4	124.2
137	13.7	27.4	41.1	54.8	68.5	82.2	95.9	109.6	123.3
136	13.6	27.2	40.8	54.4	68.0	81.6	95.2	108.8	122.4
135	13.5	27.0	40.5	54.0	67.5	81.0	94.5	108.0	121.5
134	13.4	26.8	40.2	53.6	67.0	80.4	93.8	107.2	120.6
133	13.3	26.6	39.9	53.2	66.5	79.8	93.1	106.4	119.7
132	13.2	26.4	39.6	52.8	66.0	79.2	92.4	105.6	118.8
131	13.1	26.2	39.3	52.4	65.5	78.6	91.7	104.8	117.9
130	13.0	26.0	39.0	52.0	65.0	78.0	91.0	104.0	117.0
129	12.9	25.8	38.7	51.6	64.5	77.4	90.3	103.2	116.1
128	12.8	25.6	38.4	51.2	64.0	76.8	89.6	102.4	115.2
127	12.7	25.4	38.1	50.8	63.5	76.2	88.9	101.6	114.3
126	12.6	25.2	37.8	50.4	63.0	75.6	88.2	100.8	113.4
125	12.5	25.0	37.5	50.0	62.5	75.0	87.5	100.0	112.5
124	12.4	24.8	37.2	49.6	62.0	74.4	86.8	99.2	111.6

N.	0	1	2	3	4	5	6	7	8	9	Diff.
350	54 4068	4192	4316	4440	4564	4688	4812	4936	5060	5183	124
1	5307	5431	5555	5678	5802	5925	6049	6172	6296	6419	124
2	6543	6666	6789	6913	7036	7159	7282	7405	7529	7652	123
3	7775	7898	8021	8144	8267	8389	8512	8635	8758	8881	123
4	9003	9126	9249	9371	9494	9616	9739	9861	9984	*0106	123
355	55 0228	0351	0473	0595	0717	0840	0962	1084	1206	1328	122
6	1450	1572	1694	1816	1938	2060	2181	2303	2425	2547	122
7	2668	2790	2911	3033	3155	3276	3398	3519	3640	3762	121
8	3883	4004	4126	4247	4368	4489	4610	4731	4852	4973	121
9	5094	5215	5336	5457	5578	5699	5820	5940	6061	6182	121
360	55 6303	6423	6544	6664	6785	6905	7026	7146	7267	7387	120
1	7507	7627	7748	7868	7988	8108	8228	8349	8469	8589	120
2	8709	8829	8948	9068	9188	9308	9428	9548	9667	9787	120
3	9907	*0026	*0146	*0265	*0385	*0504	*0624	*0743	*0863	*0982	119
4	56 1101	1221	1340	1459	1578	1698	1817	1936	2055	2174	119
365	56 2293	2412	2531	2650	2769	2887	3006	3125	3244	3362	119
6	3481	3600	3718	3837	3955	4074	4192	4311	4429	4548	119
7	4666	4784	4903	5021	5139	5257	5376	5494	5612	5730	118
8	5848	5966	6084	6202	6320	6437	6555	6673	6791	6909	118
9	7026	7144	7262	7379	7497	7614	7732	7849	7967	8084	118
370	56 8202	8319	8436	8554	8671	8788	8905	9023	9140	9257	117
1	9374	9491	9608	9725	9842	9959	*0076	*0193	*0309	*0426	117
2	57 0543	0660	0776	0893	1010	1126	1243	1359	1476	1592	117
3	1709	1825	1942	2058	2174	2291	2407	2523	2639	2755	116
4	2872	2988	3104	3220	3336	3452	3568	3684	3800	3915	116
375	57 4031	4147	4263	4379	4494	4610	4726	4841	4957	5072	116
6	5188	5303	5419	5534	5650	5765	5880	5996	6111	6226	115
7	6341	6457	6572	6687	6802	6917	7032	7147	7262	7377	
8	7492	7607	7722	7836	7951	8066	8181	8295	8410	8525	
9	8639	8754	8868	8983	9097	9212	9326	9441	9555	9669	
380	57 9784	9898	*0012	*0126	*0241	*0355	*0469	*0583	*0697	*0811	114
1	58 0925	1039	1153	1267	1381	1495	1608	1722	1836	1950	
2	2063	2177	2291	2404	2518	2631	2745	2858	2972	3085	
3	3199	3312	3426	3539	3652	3765	3879	3992	4105	4218	113
4	4331	4444	4557	4670	4783	4896	5009	5122	5235	5348	
385	58 5461	5574	5686	5799	5912	6024	6137	6250	6362	6475	
6	6587	6700	6812	6925	7037	7149	7262	7374	7486	7599	112
7	7711	7823	7935	8047	8160	8272	8384	8496	8608	8720	
8	8832	8944	9056	9167	9279	9391	9503	9615	9726	9838	
9	9950	*0061	*0173	*0284	*0396	*0507	*0619	*0730	*0842	*0953	

PROPORTIONAL PARTS

Diff.	1	2	3	4	5	6	7	8	9
124	12.4	24.8	37.2	49.6	62.0	74.4	86.8	99.2	111.6
123	12.3	24.6	36.9	49.2	61.5	73.8	86.1	98.4	110.7
122	12.2	24.4	36.6	48.8	61.0	73.2	85.4	97.6	109.8
121	12.1	24.2	36.3	48.4	60.5	72.6	84.7	96.8	108.9
120	12.0	24.0	36.0	48.0	60.0	72.0	84.0	96.0	108.0
119	11.9	23.8	35.7	47.6	59.5	71.4	83.3	95.2	107.1
118	11.8	23.6	35.4	47.2	59.0	70.8	82.6	94.4	106.2
117	11.7	23.4	35.1	46.8	58.5	70.2	81.9	93.6	105.3
116	11.6	23.2	34.8	46.4	58.0	69.6	81.2	92.8	104.4
115	11.5	23.0	34.5	46.0	57.5	69.0	80.5	92.0	103.5
114	11.4	22.8	34.2	45.6	57.0	68.4	79.8	91.2	102.6
113	11.3	22.6	33.9	45.2	56.5	67.8	79.1	90.4	101.7
112	11.2	22.4	33.6	44.8	56.0	67.2	78.4	89.6	100.8

Table XV—Six-Place Logarithms

N.	0	1	2	3	4	5	6	7	8	9	Diff
390	59 1065	1176	1287	1399	1510	1621	1732	1843	1955	2066	111
1	2177	2288	2399	2510	2621	2732	2843	2954	3064	3175	
2	3286	3397	3508	3618	3729	3840	3950	4061	4171	4282	
3	4393	4503	4614	4724	4834	4945	5055	5165	5276	5386	110
4	5496	5606	5717	5827	5937	6047	6157	6267	6377	6487	
395	59 6597	6707	6817	6927	7037	7146	7256	7366	7476	7586	
6	7695	7805	7914	8024	8134	8243	8353	8462	8572	8681	
7	8791	8900	9009	9119	9228	9337	9446	9556	9665	9774	109
8	9883	9992	*0101	*0210	*0319	*0428	*0537	*0646	*0755	*0864	
9	60 0973	1082	1191	1299	1408	1517	1625	1734	1843	1951	
400	60 2060	2169	2277	2386	2494	2603	2711	2819	2928	3036	108
1	3144	3253	3361	3469	3577	3686	3794	3902	4010	4118	
2	4226	4334	4442	4550	4658	4766	4874	4982	5089	5197	
3	5305	5413	5521	5628	5736	5844	5951	6059	6166	6274	
4	6381	6489	6596	6704	6811	6919	7026	7133	7241	7348	107
405	60 7455	7562	7669	7777	7884	7991	8098	8205	8312	8419	
6	8526	8633	8740	8847	8954	9061	9167	9274	9381	9488	
7	9594	9701	9808	9914	*0021	*0128	*0234	*0341	*0447	*0554	
8	61 0660	0767	0873	0979	1086	1192	1298	1405	1511	1617	106
9	1723	1829	1936	2042	2148	2254	2360	2466	2572	2678	
410	61 2784	2890	2996	3102	3207	3313	3419	3525	3630	3736	
1	3842	3947	4053	4159	4264	4370	4475	4581	4686	4792	
2	4897	5003	5108	5213	5319	5424	5529	5634	5740	5845	105
3	5950	6055	6160	6265	6370	6476	6581	6686	6790	6895	
4	7000	7105	7210	7315	7420	7525	7629	7734	7839	7943	
415	61 8048	8153	8257	8362	8466	8571	8676	8780	8884	8989	
6	9093	9198	9302	9406	9511	9615	9719	9824	9928	*0032	104
7	62 0136	0240	0344	0448	0552	0656	0760	0864	0968	1072	
8	1176	1280	1384	1488	1592	1695	1799	1903	2007	2110	
9	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146	
420	62 3249	3353	3456	3559	3663	3766	3869	3973	4076	4179	103
1	4282	4385	4488	4591	4695	4798	4901	5004	5107	5210	
2	5312	5415	5518	5621	5724	5827	5929	6032	6135	6238	
3	6340	6443	6546	6648	6751	6853	6956	7058	7161	7263	
4	7366	7468	7571	7673	7775	7878	7980	8082	8185	8287	102
425	62 8389	8491	8593	8695	8797	8900	9002	9104	9206	9308	
6	9410	9512	9613	9715	9817	9919	*0021	*0123	*0224	*0326	
7	63 0428	0530	0631	0733	0835	0936	1038	1139	1241	1342	
8	1444	1545	1647	1748	1849	1951	2052	2153	2255	2356	101
9	2457	2559	2660	2761	2862	2963	3064	3165	3266	3367	

PROPORTIONAL PARTS

Diff.	1	2	3	4	5	6	7	8	9
111	11.1	22.2	33.3	44.4	55.5	66.6	77.7	88.8	99.9
110	11.0	22.0	33.0	44.0	55.0	66.0	77.0	88.0	99.0
109	10.9	21.8	32.7	43.6	54.5	65.4	76.3	87.2	98.1
108	10.8	21.6	32.4	43.2	54.0	64.8	75.6	86.4	97.2
107	10.7	21.4	32.1	42.8	53.5	64.2	74.9	85.6	96.3
106	10.6	21.2	31.8	42.4	53.0	63.6	74.2	84.8	95.4
105	10.5	21.0	31.5	42.0	52.5	63.0	73.5	84.0	94.5
104	10.4	20.8	31.2	41.6	52.0	62.4	72.8	83.2	93.6
103	10.3	20.6	30.9	41.2	51.5	61.8	72.1	82.4	92.7
102	10.2	20.4	30.6	40.8	51.0	61.2	71.4	81.6	91.8
101	10.1	20.2	30.3	40.4	50.5	60.6	70.7	80.8	90.9

N.	0	1	2	3	4	5	6	7	8	9	Diff.
430	63 3468	3569	3670	3771	3872	3973	4074	4175	4276	4376	
1	4477	4578	4679	4779	4880	4981	5081	5182	5283	5383	
2	5484	5584	5685	5785	5886	5986	6087	6187	6287	6388	100
3	6488	6588	6688	6789	6889	6989	7089	7189	7290	7390	
4	7490	7590	7690	7790	7890	7990	8090	8190	8290	8389	
435	63 8489	8589	8689	8789	8888	8988	9088	9188	9287	9387	
6	9486	9586	9686	9785	9885	9984	*0084	*0183	*0283	*0382	
7	64 0481	0581	0680	0779	0879	0978	1077	1177	1276	1375	99
8	1474	1573	1672	1771	1871	1970	2069	2168	2267	2366	
9	2465	2563	2662	2761	2860	2959	3058	3156	3255	3354	
440	64 3453	3551	3650	3749	3847	3946	4044	4143	4242	4340	
1	4439	4537	4636	4734	4832	4931	5029	5127	5226	5324	98
2	5422	5521	5619	5717	5815	5913	6011	6110	6208	6306	
3	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285	
4	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262	
445	64 8360	8458	8555	8653	8750	8848	8945	9043	9140	9237	
6	9335	9432	9530	9627	9724	9821	9919	*0016	*0113	*0210	97
7	65 0308	0405	0502	0599	0696	0793	0890	0987	1084	1181	
8	1278	1375	1472	1569	1666	1762	1859	1956	2053	2150	
9	2246	2343	2440	2536	2633	2730	2826	2923	3019	3116	
450	65 3213	3309	3405	3502	3598	3695	3791	3888	3984	4080	96
1	4177	4273	4369	4465	4562	4658	4754	4850	4946	5042	
2	5138	5235	5331	5427	5523	5619	5715	5810	5906	6002	
3	6098	6194	6290	6386	6482	6577	6673	6769	6864	6960	
4	7056	7152	7247	7343	7438	7534	7629	7725	7820	7916	
455	65 8011	8107	8202	8298	8393	8488	8584	8679	8774	8870	95
6	8965	9060	9155	9250	9346	9441	9536	9631	9726	9821	
7	9916	*0011	*0106	*0201	*0296	*0391	*0486	*0581	*0676	*0771	
8	66 0865	0960	1055	1150	1245	1339	1434	1529	1623	1718	
9	1813	1907	2002	2096	2191	2286	2380	2475	2569	2663	
460	66 2758	2852	2947	3041	3135	3230	3324	3418	3512	3607	94
1	3701	3795	3889	3983	4078	4172	4266	4360	4454	4548	
2	4642	4736	4830	4924	5018	5112	5206	5299	5393	5487	
3	5581	5675	5769	5862	5956	6050	6143	6237	6331	6424	
4	6518	6612	6705	6799	6892	6986	7079	7173	7266	7360	
465	66 7453	7546	7640	7733	7826	7920	8013	8106	8199	8293	93
6	8386	8479	8572	8665	8759	8852	8945	9038	9131	9224	
7	9317	9410	9503	9596	9689	9782	9875	9967	*0060	*0153	
8	67 0246	0339	0431	0524	0617	0710	0802	0895	0988	1080	
9	1173	1265	1358	1451	1543	1636	1728	1821	1913	2005	

PROPORTIONAL PARTS

Diff.	1	2	3	4	5	6	7	8	9
100	10.0	20.0	30.0	40.0	50.0	60.0	70.0	80.0	90.0
99	9.9	19.8	29.7	39.6	49.5	59.4	69.3	79.2	89.1
98	9.8	19.6	29.4	39.2	49.0	58.8	68.6	78.4	88.2
97	9.7	19.4	29.1	38.8	48.5	58.2	67.9	77.6	87.3
96	9.6	19.2	28.8	38.4	48.0	57.6	67.2	76.8	86.4
95	9.5	19.0	28.5	38.0	47.5	57.0	66.5	76.0	85.5
94	9.4	18.8	28.2	37.6	47.0	56.4	65.8	75.2	84.6
93	9.3	18.6	27.9	37.2	46.5	55.8	65.1	74.4	83.7
92	9.2	18.4	27.6	36.8	46.0	55.2	64.4	73.6	82.8

Table XV—Six-Place Logarithms

N.	0	1	2	3	4	5	6	7	8	9	Diff.
470	67 2098	2190	2283	2375	2467	2560	2652	2744	2836	2929	92
1	3021	3113	3205	3297	3390	3482	3574	3666	3758	3850	
2	3942	4034	4126	4218	4310	4402	4494	4586	4677	4769	
3	4861	4953	5045	5137	5228	5320	5412	5503	5595	5687	
4	5778	5870	5962	6053	6145	6236	6328	6419	6511	6602	
475	67 6694	6785	6876	6968	7059	7151	7242	7333	7424	7516	91
6	7607	7698	7789	7881	7972	8063	8154	8245	8336	8427	
7	8518	8609	8700	8791	8882	8973	9064	9155	9246	9337	
8	9428	9519	9610	9700	9791	9882	9973	*0063	*0154	*0245	
9	68 0336	0426	0517	0607	0698	0789	0879	0970	1060	1151	
480	68 1241	1332	1422	1513	1603	1693	1784	1874	1964	2055	90
1	2145	2235	2326	2416	2506	2596	2686	2777	2867	2957	
2	3047	3137	3227	3317	3407	3497	3587	3677	3767	3857	
3	3947	4037	4127	4217	4307	4396	4486	4576	4666	4756	
4	4845	4935	5025	5114	5204	5294	5383	5473	5563	5652	
485	68 5742	5831	5921	6010	6100	6189	6279	6368	6458	6547	89
6	6636	6726	6815	6904	6994	7083	7172	7261	7351	7440	
7	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331	
8	8420	8509	8598	8687	8776	8865	8953	9042	9131	9220	
9	9309	9398	9486	9575	9664	9753	9841	9930	*0019	*0107	
490	69 0196	0285	0373	0462	0550	0639	0728	0816	0905	0993	
1	1081	1170	1258	1347	1435	1524	1612	1700	1789	1877	88
2	1965	2053	2142	2230	2318	2406	2494	2583	2671	2759	
3	2847	2935	3023	3111	3199	3287	3375	3463	3551	3639	
4	3727	3815	3903	3991	4078	4166	4254	4342	4430	4517	
495	69 4605	4693	4781	4868	4956	5044	5131	5219	5307	5394	
6	5482	5569	5657	5744	5832	5919	6007	6094	6182	6269	87
7	6356	6444	6531	6618	6706	6793	6880	6968	7055	7142	
8	7229	7317	7404	7491	7578	7665	7752	7839	7926	8014	
9	8101	8188	8275	8362	8449	8535	8622	8709	8796	8883	
500	69 8970	9057	9144	9231	9317	9404	9491	9578	9664	9751	
1	9838	9924	*0011	*0098	*0184	*0271	*0358	*0444	*0531	*0617	
2	70 0704	0790	0877	0963	1050	1136	1222	1309	1395	1482	86
3	1568	1654	1741	1827	1913	1999	2086	2172	2258	2344	
4	2431	2517	2603	2689	2775	2861	2947	3033	3119	3205	
505	70 3291	3377	3463	3549	3635	3721	3807	3893	3979	4065	
6	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922	
7	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778	
8	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632	85
9	6718	6803	6888	6974	7059	7144	7229	7315	7400	7485	
510	70 7570	7655	7740	7826	7911	7996	8081	8166	8251	8336	
1	8421	8506	8591	8676	8761	8846	8931	9015	9100	9185	
2	9270	9355	9440	9524	9609	9694	9779	9863	9948	*0033	
3	71 0117	0202	0287	0371	0456	0540	0625	0710	0794	0879	
4	0963	1048	1132	1217	1301	1385	1470	1554	1639	1723	

PROPORTIONAL PARTS

Diff	1	2	3	4	5	6	7	8	9
91	9.1	18.2	27.3	36.4	45.5	54.6	63.7	72.8	81.9
90	9.0	18.0	27.0	36.0	45.0	54.0	63.0	72.0	81.0
89	8.9	17.8	26.7	35.6	44.5	53.4	62.3	71.2	80.1
88	8.8	17.6	26.4	35.2	44.0	52.8	61.6	70.4	79.2
87	8.7	17.4	26.1	34.8	43.5	52.2	60.9	69.6	78.3
86	8.6	17.2	25.8	34.4	43.0	51.6	60.2	68.8	77.4
85	8.5	17.0	25.5	34.0	42.5	51.0	59.5	68.0	76.5
84	8.4	16.8	25.2	33.6	42.0	50.4	58.8	67.2	75.6

N.	0	1	2	3	4	5	6	7	8	9	Diff.
515	71 1807	1892	1976	2060	2144	2229	2313	2397	2481	2566	
6	2650	2734	2818	2902	2986	3070	3154	3238	3323	3407	84
7	3491	3575	3659	3742	3826	3910	3994	4078	4162	4246	
8	4330	4414	4497	4581	4665	4749	4833	4916	5000	5084	
9	5167	5251	5335	5418	5502	5586	5669	5753	5836	5920	
520	71 6003	6087	6170	6254	6337	6421	6504	6588	6671	6754	
1	6838	6921	7004	7088	7171	7254	7338	7421	7504	7587	83
2	7671	7754	7837	7920	8003	8086	8169	8253	8336	8419	
3	8502	8585	8668	8751	8834	8917	9000	9083	9165	9248	
4	9331	9414	9497	9580	9663	9745	9828	9911	9994	*0077	
525	72 0159	0242	0325	0407	0490	0573	0655	0738	0821	0903	
6	0986	1068	1151	1233	1316	1398	1481	1563	1646	1728	
7	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552	82
8	2634	2716	2798	2881	2963	3045	3127	3209	3291	3374	
9	3456	3538	3620	3702	3784	3866	3948	4030	4112	4194	
530	72 4276	4358	4440	4522	4604	4685	4767	4849	4931	5013	
1	5095	5176	5258	5340	5422	5503	5585	5667	5748	5830	
2	5912	5993	6075	6156	6238	6320	6401	6483	6564	6646	
3	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460	81
4	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273	
535	72 8354	8435	8516	8597	8678	8759	8841	8922	9003	9084	
6	9165	9246	9327	9408	9489	9570	9651	9732	9813	9893	
7	9974	*0055	*0136	*0217	*0298	*0378	*0459	*0540	*0621	*0702	
8	73 0782	0863	0944	1024	1105	1186	1266	1347	1428	1508	
9	1589	1669	1750	1830	1911	1991	2072	2152	2233	2313	
540	73 2394	2474	2555	2635	2715	2796	2876	2956	3037	3117	80
1	3197	3278	3358	3438	3518	3598	3679	3759	3839	3919	
2	3999	4079	4160	4240	4320	4400	4480	4560	4640	4720	
3	4800	4880	4960	5040	5120	5200	5279	5359	5439	5519	
4	5599	5679	5759	5838	5918	5998	6078	6157	6237	6317	
545	73 6397	6476	6556	6635	6715	6795	6874	6954	7034	7113	
6	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908	79
7	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701	
8	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493	
9	9572	9651	9731	9810	9889	9968	*0047	*0126	*0205	*0284	
550	74 0363	0442	0521	0600	0678	0757	0836	0915	0994	1073	
1	1152	1230	1309	1388	1467	1546	1624	1703	1782	1860	
2	1939	2018	2096	2175	2254	2332	2411	2489	2568	2647	78
3	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431	
4	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215	

PROPORTIONAL PARTS

Diff.	1	2	3	4	5	6	7	8	9
85	8.5	17.0	25.5	34.0	42 5	51.0	59.5	68.0	76.5
84	8.4	16.8	25.2	33.6	42.0	50.4	58.8	67 2	75.6
83	8.3	16.6	24.9	33.2	41.5	49.8	58.1	66.4	74.7
82	8.2	16.4	24.6	32.8	41.0	49.2	57.4	65.6	73.8
81	8.1	16.2	24.3	32.4	40.5	48.6	56.7	64.8	72.9
80	8.0	16.0	24.0	32.0	40.0	48.0	56.0	64.0	72.0
79	7.9	15.8	23.7	31.6	39.5	47.4	55.3	63.2	71.1
78	7.8	15.6	23.4	31.2	39.0	46.8	54.6	62.4	70.2

Table XV—Six-Place Logarithms

N.	0	1	2	3	4	5	6	7	8	9	Diff.
555	74 4293	4371	4449	4528	4606	4684	4762	4840	4919	4997	
6	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777	
7	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556	
8	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334	
9	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110	
560	74 8188	8266	8343	8421	8498	8576	8653	8731	8808	8885	
1	8963	9040	9118	9195	9272	9350	9427	9504	9582	9659	77
2	9736	9814	9891	9968	*0045	*0123	*0200	*0277	*0354	*0431	
3	75 0508	0586	0663	0740	0817	0894	0971	1048	1125	1202	
4	1279	1356	1433	1510	1587	1664	1741	1818	1895	1972	
565	75 2048	2125	2202	2279	2356	2433	2509	2586	2663	2740	
6	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506	
7	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272	
8	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036	76
9	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799	
570	75 5875	5951	6027	6103	6180	6256	6332	6408	6484	6560	
1	6636	6712	6788	6864	6940	7016	7092	7168	7244	7320	
2	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079	
3	8155	8230	8306	8382	8458	8533	8609	8685	8761	8836	
4	8912	8988	9063	9139	9214	9290	9366	9441	9517	9592	
575	75 9668	9743	9819	9894	9970	*0045	*0121	*0196	*0272	*0347	75
6	76 0422	0498	0573	0649	0724	0799	0875	0950	1025	1101	
7	1176	1251	1326	1402	1477	1552	1627	1702	1778	1853	
8	1928	2003	2078	2153	2228	2303	2378	2453	2529	2604	
9	2679	2754	2829	2904	2978	3053	3128	3203	3278	3353	
580	76 3428	3503	3578	3653	3727	3802	3877	3952	4027	4101	
1	4176	4251	4326	4400	4475	4550	4624	4699	4774	4848	
2	4923	4998	5072	5147	5221	5296	5370	5445	5520	5594	
3	5669	5743	5818	5892	5966	6041	6115	6190	6264	6338	74
4	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082	
585	76 7156	7230	7304	7379	7453	7527	7601	7675	7749	7823	
6	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564	
7	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303	
8	9377	9451	9525	9599	9673	9746	9820	9894	9968	*0042	
9	77 0115	0189	0263	0336	0410	0484	0557	0631	0705	0778	
590	77 0852	0926	0999	1073	1146	1220	1293	1367	1440	1514	
1	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248	
2	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981	73
3	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713	
4	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444	
595	77 4517	4590	4663	4736	4809	4882	4955	5028	5100	5173	
6	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902	
7	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629	
8	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354	
9	7427	7499	7572	7644	7717	7789	7862	7934	8006	8079	

PROPORTIONAL PARTS

Diff.	1	2	3	4	5	6	7	8	9
77	7.7	15.4	23.1	30.8	38.5	46.2	53.9	61 6	69.3
76	7.6	15.2	22.8	30.4	38.0	45.6	53.2	60.8	68.4
75	7.5	15.0	22.5	30.0	37.5	45.0	52.5	60.0	67.5
74	7.4	14.8	22.2	29.6	37.0	44.4	51.8	59 2	66 6
73	7.3	14.6	21.9	29.2	36.5	43.8	51.1	58.4	65 7
72	7.2	14.4	21.6	28.8	36.0	43.2	50.4	57.6	64.8
71	7.1	14.2	21.3	28.4	35.5	42.6	49.7	56 8	63.9

N.	0	1	2	3	4	5	6	7	8	9	Diff.
600	77 8151	8224	8296	8368	8441	8513	8585	8658	8730	8802	72
1	8874	8947	9019	9091	9163	9236	9308	9380	9452	9524	
2	9596	9669	9741	9813	9885	9957	*0029	*0101	*0173	*0245	
3	78 0317	0389	0461	0533	0605	0677	0749	0821	0893	0965	
4	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684	
605	78 1755	1827	1899	1971	2042	2114	2186	2258	2329	2401	
6	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117	
7	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832	
8	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546	71
9	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259	
610	78 5330	5401	5472	5543	5615	5686	5757	5828	5899	5970	
1	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680	
2	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390	
3	7460	7531	7602	7673	7744	7815	7885	7956	8027	8098	
4	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804	
615	78 8875	8946	9016	9087	9157	9228	9299	9369	9440	9510	
6	9581	9651	9722	9792	9863	9933	*0004	*0074	*0144	*0215	70
7	79 0285	0356	0426	0496	0567	0637	0707	0778	0848	0918	
8	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620	
9	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322	
620	79 2392	2462	2532	2602	2672	2742	2812	2882	2952	3022	
1	3092	3162	3231	3301	3371	3441	3511	3581	3651	3721	
2	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418	
3	4488	4558	4627	4697	4767	4836	4906	4976	5045	5115	
4	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811	
625	79 5880	5949	6019	6088	6158	6227	6297	6366	6436	6505	
6	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198	69
7	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890	
8	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582	
9	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272	
630	79 9341	9409	9478	9547	9616	9685	9754	9823	9892	9961	
1	80 0029	0098	0167	0236	0305	0373	0442	0511	0580	0648	
2	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335	
3	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021	
4	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705	
635	80 2774	2842	2910	2979	3047	3116	3184	3252	3321	3389	68
6	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071	
7	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753	
8	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433	
9	5501	5569	5637	5705	5773	5841	5908	5976	6044	6112	
640	80 6180	6248	6316	6384	6451	6519	6587	6655	6723	6790	
1	6858	6926	6994	7061	7129	7197	7264	7332	7400	7467	
2	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143	
3	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818	
4	8886	8953	9021	9088	9156	9223	9290	9358	9425	9492	67
645	80 9560	9627	9694	9762	9829	9896	9964	*0031	*0098	*0165	
6	81 0233	0300	0367	0434	0501	0569	0636	0703	0770	0837	
7	0904	0971	1039	1106	1173	1240	1307	1374	1441	1508	
8	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178	
9	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847	

PROPORTIONAL PARTS

Diff.	1	2	3	4	5	6	7	8	9
70	7.0	14.0	21.0	28.0	35.0	42.0	49.0	56.0	63.0
69	6.9	13.8	20.7	27.6	34.5	41.4	48.3	55.2	62.1
68	6.8	13.6	20.4	27.2	34.0	40.8	47.6	54.4	61.2
67	6.7	13.4	20.1	26.8	33.5	40.2	46.9	53.6	60.3

Table XV—Six-Place Logarithms

N.	0	1	2	3	4	5	6	7	8	9	Diff
650	81 2913	2980	3047	3114	3181	3247	3314	3381	3448	3514	
1	3581	3648	3714	3781	3848	3914	3981	4048	4114	4181	
2	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847	
3	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511	
4	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175	66
655	81 6241	6308	6374	6440	6506	6573	6639	6705	6771	6838	
6	6904	6970	7036	7102	7169	7235	7301	7367	7433	7499	
7	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160	
8	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820	
9	8885	8951	9017	9083	9149	9215	9281	9346	9412	9478	
660	81 9544	9610	9676	9741	9807	9873	9939	*0004	*0070	*0136	
1	82 0201	0267	0333	0399	0464	0530	0595	0661	0727	0792	
2	0858	0924	0989	1055	1120	1186	1251	1317	1382	1448	
3	1514	1579	1645	1710	1775	1841	1906	1972	2037	2103	65
4	2168	2233	2299	2364	2430	2495	2560	2626	2691	2756	
665	82 2822	2887	2952	3018	3083	3148	3213	3279	3344	3409	
6	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061	
7	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711	
8	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361	
9	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010	
670	82 6075	6140	6204	6269	6334	6399	6464	6528	6593	6658	
1	6723	6787	6852	6917	6981	7046	7111	7175	7240	7305	
2	7369	7434	7499	7563	7628	7692	7757	7821	7886	7951	
3	8015	8080	8144	8209	8273	8338	8402	8467	8531	8595	
4	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239	64
675	82 9304	9368	9432	9497	9561	9625	9690	9754	9818	9882	
6	9947	*0011	*0075	*0139	*0204	*0268	*0332	*0396	*0460	*0525	
7	83 0589	0653	0717	0781	0845	0909	0973	1037	1102	1166	
8	1230	1294	1358	1422	1486	1550	1614	1678	1742	1806	
9	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445	
680	83 2509	2573	2637	2700	2764	2828	2892	2956	3020	3083	
1	3147	3211	3275	3338	3402	3466	3530	3593	3657	3721	
2	3784	3848	3912	3975	4039	4103	4166	4230	4294	4357	
3	4421	4484	4548	4611	4675	4739	4802	4866	4929	4993	
4	5056	5120	5183	5247	5310	5373	5437	5500	5564	5627	
685	83 5691	5754	5817	5881	5944	6007	6071	6134	6197	6261	63
6	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894	
7	6957	7020	7083	7146	7210	7273	7336	7399	7462	7525	
8	7588	7652	7715	7778	7841	7904	7967	8030	8093	8156	
9	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786	
690	83 8849	8912	8975	9038	9101	9164	9227	9289	9352	9415	
1	9478	9541	9604	9667	9729	9792	9855	9918	9981	*0043	
2	84 0106	0169	0232	0294	0357	0420	0482	0545	0608	0671	
3	0733	0796	0859	0921	0984	1046	1109	1172	1234	1297	
4	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922	
695	84 1985	2047	2110	2172	2235	2297	2360	2422	2484	2547	62
6	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170	
7	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793	
8	3855	3918	3980	4042	4104	4166	4229	4291	4353	4415	
9	4477	4539	4601	4664	4726	4788	4850	4912	4974	5036	

PROPORTIONAL PARTS

Diff.	1	2	3	4	5	6	7	8	9
66	6.6	13.2	19.8	26.4	33.0	39.6	46.2	52.8	59.4
65	6.5	13.0	19.5	26.0	32.5	39.0	45.5	52.0	58.5
64	6.4	12.8	19.2	25.6	32.0	38.4	44.8	51.2	57.6
63	6.3	12.6	18.9	25.2	31.5	37.8	44.1	50.4	56.7

N.	0	1	2	3	4	5	6	7	8	9	Diff.
700	84 5098	5160	5222	5284	5346	5408	5470	5532	5594	5656	62
1	5718	5780	5842	5904	5966	6028	6090	6151	6213	6275	
2	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894	
3	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511	
4	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128	
705	84 8189	8251	8312	8374	8435	8497	8559	8620	8682	8743	
6	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358	61
7	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972	
8	85 0033	0095	0156	0217	0279	0340	0401	0462	0524	0585	
9	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197	
710	85 1258	1320	1381	1442	1503	1564	1625	1686	1747	1809	
1	1870	1931	1992	2053	2114	2175	2236	2297	2358	2419	
2	2480	2541	2602	2663	2724	2785	2846	2907	2968	3029	
3	3090	3150	3211	3272	3333	3394	3455	3516	3577	3637	
4	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245	
715	85 4306	4367	4428	4488	4549	4610	4670	4731	4792	4852	
6	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459	
7	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064	
8	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668	
9	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272	60
720	85 7332	7393	7453	7513	7574	7634	7694	7755	7815	7875	
1	7935	7995	8056	8116	8176	8236	8297	8357	8417	8477	
2	8537	8597	8657	8718	8778	8838	8898	8958	9018	9078	
3	9138	9198	9258	9318	9379	9439	9499	9559	9619	9679	
4	9739	9799	9859	9918	9978	*0038	*0098	*0158	*0218	*0278	
725	86 0338	0398	0458	0518	0578	0637	0697	0757	0817	0877	
6	0937	0996	1056	1116	1176	1236	1295	1355	1415	1475	
7	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072	
8	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668	
9	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263	
730	86 3323	3382	3442	3501	3561	3620	3680	3739	3799	3858	59
1	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452	
2	4511	4570	4630	4689	4748	4808	4867	4926	4985	5045	
3	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637	
4	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228	
735	86 6287	6346	6405	6465	6524	6583	6642	6701	6760	6819	
6	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409	
7	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998	
8	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586	
9	8644	8703	8762	8821	8879	8938	8997	9056	9114	9173	
740	86 9232	9290	9349	9408	9466	9525	9584	9642	9701	9760	
1	9818	9877	9935	9994	*0053	*0111	*0170	*0228	*0287	*0345	
2	87 0404	0462	0521	0579	0638	0696	0755	0813	0872	0930	
3	0989	1047	1106	1164	1223	1281	1339	1398	1456	1515	58
4	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098	
745	87 2156	2215	2273	2331	2389	2448	2506	2564	2622	2681	
6	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262	
7	3321	3379	3437	3495	3553	3611	3669	3727	3785	3844	
8	3902	3960	4018	4076	4134	4192	4250	4308	4366	4424	
9	4482	4540	4598	4656	4714	4772	4830	4888	4945	5003	

PROPORTIONAL PARTS

Diff.	1	2	3	4	5	6	7	8	9
62	6.2	12.4	18.6	24.8	31.0	37.2	43.4	49.6	55.8
61	6.1	12.2	18 3	24.4	30.5	36.6	42.7	48.8	54.9
60	6.0	12.0	18.0	24.0	30.0	36.0	42.0	48.0	54.0
59	5.9	11.8	17.7	23.6	29.5	35.4	41.3	47.2	53.1

N.	0	1	2	3	4	5	6	7	8	9	Diff.
750	87 5061	5119	5177	5235	5293	5351	5409	5466	5524	5582	
1	5640	5698	5756	5813	5871	5929	5987	6045	6102	6160	
2	6218	6276	6333	6391	6449	6507	6564	6622	6680	6737	
3	6795	6853	6910	6968	7026	7083	7141	7199	7256	7314	
4	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889	
755	87 7947	8004	8062	8119	8177	8234	8292	8349	8407	8464	
6	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039	57
7	9096	9153	9211	9268	9325	9383	9440	9497	9555	9612	
8	9669	9726	9784	9841	9898	9956	*0013	*0070	*0127	*0185	
9	88 0242	0299	0356	0413	0471	0528	0585	0642	0699	0756	
760	88 0814	0871	0928	0985	1042	1099	1156	1213	1271	1328	
1	1385	1442	1499	1556	1613	1670	1727	1784	1841	1898	
2	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468	
3	2525	2581	2638	2695	2752	2809	2866	2923	2980	3037	
4	3093	3150	3207	3264	3321	3377	3434	3491	3548	3605	
765	88 3661	3718	3775	3832	3888	3945	4002	4059	4115	4172	
6	4229	4285	4342	4399	4455	4512	4569	4625	4682	4739	
7	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305	
8	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870	
9	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434	
770	88 3491	6547	6604	6660	6716	6773	6829	6885	6942	6998	56
1	7054	7111	7167	7223	7280	7336	7392	7449	7505	7561	
2	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123	
3	8179	8236	8292	8348	8404	8460	8516	8573	8629	8685	
4	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246	
775	88 9302	9358	9414	9470	9526	9582	9638	9694	9750	9806	
6	9862	9918	9974	*0030	*0086	*0141	*0197	*0253	*0309	*0365	
7	89 0421	0477	0533	0589	0645	0700	0756	0812	0868	0924	
8	0980	1035	1091	1147	1203	1259	1314	1370	1426	1482	
9	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039	
780	89 2095	2150	2206	2262	2317	2373	2429	2484	2540	2595	
1	2651	2707	2762	2818	2873	2929	2985	3040	3096	3151	
2	3207	3262	3318	3373	3429	3484	3540	3595	3651	3706	
3	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261	55
4	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814	
785	89 4870	4925	4980	5036	5091	5146	5201	5257	5312	5367	
6	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920	
7	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471	
8	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022	
9	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572	
790	89 7627	7682	7737	7792	7847	7902	7957	8012	8067	8122	
1	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670	
2	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218	
3	9273	9328	9383	9437	9492	9547	9602	9656	9711	9766	
4	9821	9875	9930	9985	*0039	*0094	*0149	*0203	*0258	*0312	
795	90 0367	0422	0476	0531	0586	0640	0695	0749	0804	0859	
6	0913	0968	1022	1077	1131	1186	1240	1295	1349	1404	
7	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948	
8	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492	54
9	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036	

PROPORTIONAL PARTS

Diff.	1	2	3	4	5	6	7	8	9
58	5.8	11.6	17.4	23.2	29.0	34.8	40.6	46.4	52.2
57	5.7	11.4	17.1	22.8	28.5	34.2	39.9	45.6	51 3
56	5.6	11.2	16.8	22.4	28.0	33.6	39.2	44.8	50.4
55	5.5	11.0	16.5	22.0	27 5	33 0	38.5	44 0	49.5

N.	0	1	2	3	4	5	6	7	8	9	Diff.
800	90 3090	3144	3199	3253	3307	3361	3416	3470	3524	3578	54
1	3633	3687	3741	3795	3849	3904	3958	4012	4066	4120	
2	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661	
3	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202	
4	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742	
805	90 5796	5850	5904	5958	6012	6066	6119	6173	6227	6281	
6	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820	
7	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358	
8	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895	
9	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431	
810	90 8485	8539	8592	8646	8699	8753	8807	8860	8914	8967	
1	9021	9074	9128	9181	9235	9289	9342	9396	9449	9503	
2	9556	9610	9663	9716	9770	9823	9877	9930	9984	*0037	
3	91 0091	0144	0197	0251	0304	0358	0411	0464	0518	0571	53
4	0624	0678	0731	0784	0838	0891	0944	0998	1051	1104	
815	91 1158	1211	1264	1317	1371	1424	1477	1530	1584	1637	
6	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169	
7	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700	
8	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231	
9	3284	3337	3390	3443	3496	3549	3602	3655	3708	3761	
820	91 3814	3867	3920	3973	4026	4079	4132	4184	4237	4290	
1	4343	4396	4449	4502	4555	4608	4660	4713	4766	4819	
2	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347	
3	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875	
4	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401	
825	91 6454	6507	6559	6612	6664	6717	6770	6822	6875	6927	
6	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453	
7	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978	
8	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502	
9	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026	52
830	91 9078	9130	9183	9235	9287	9340	9392	9444	9496	9549	
1	9601	9653	9706	9758	9810	9862	9914	9967	*0019	*0071	
2	92 0123	0176	0228	0280	0332	0384	0436	0489	0541	0593	
3	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114	
4	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634	
835	92 1686	1738	1790	1842	1894	1946	1998	2050	2102	2154	
6	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674	
7	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192	
8	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710	
9	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228	
840	92 4279	4331	4383	4434	4486	4538	4589	4641	4693	4744	
1	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261	
2	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776	
3	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291	51
4	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805	
845	92 6857	6908	6959	7011	7062	7114	7165	7216	7268	7319	
6	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832	
7	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345	
8	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857	
9	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368	

PROPORTIONAL PARTS

Diff.	1	2	3	4	5	6	7	8	9
55	5.5	11.0	16.5	22.0	27.5	33.0	38.5	44.0	49.5
54	5.4	10.8	16.2	21.6	27.0	32.4	37.8	43.2	48.6
53	5.3	10.6	15.9	21.2	26.5	31.8	37.1	42.4	47.7
52	5.2	10.4	15.6	20.8	26.0	31.2	36.4	41.6	46.8

Table XV—Six-Place Logarithms

N.	0	1	2	3	4	5	6	7	8	9	Diff.
850	92 9419	9470	9521	9572	9623	9674	9725	9776	9827	9879	
1	9930	9981	*0032	*0083	*0134	*0185	*0236	*0287	*0338	*0389	51
2	93 0440	0491	0542	0592	0643	0694	0745	0796	0847	0898	
3	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407	
4	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915	
855	93 1966	2017	2068	2118	2169	2220	2271	2322	2372	2423	
6	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930	
7	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437	
8	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943	
9	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448	
860	93 4498	4549	4599	4650	4700	4751	4801	4852	4902	4953	
1	5003	5054	5104	5154	5205	5255	5306	5356	5406	5457	50
2	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960	
3	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463	
4	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966	
865	93 7016	7066	7117	7167	7217	7267	7317	7367	7418	7468	
6	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969	
7	8019	8069	8119	8169	8219	8269	8320	8370	8420	8470	
8	8520	8570	8620	8670	8720	8770	8820	8870	8920	8970	
9	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469	
870	93 9519	9569	9619	9669	9719	9769	9819	9869	9918	9968	
1	94 0018	0068	0118	0168	0218	0267	0317	0367	0417	0467	
2	0516	0566	0616	0666	0716	0765	0815	0865	0915	0964	
3	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462	
4	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958	
875	94 2008	2058	2107	2157	2207	2256	2306	2355	2405	2455	
6	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950	
7	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445	
8	3495	3544	3593	3643	3692	3742	3791	3841	3890	3939	49
9	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433	
880	94 4483	4532	4581	4631	4680	4729	4779	4828	4877	4927	
1	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419	
2	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912	
3	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403	
4	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894	
885	94 6943	6992	7041	7090	7140	7189	7238	7287	7336	7385	
6	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875	
7	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364	
8	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853	
9	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341	
890	94 9390	9439	9488	9536	9585	9634	9683	9731	9780	9829	
1	9878	9926	9975	*0024	*0073	*0121	*0170	*0219	*0267	*0316	
2	95 0365	0414	0462	0511	0560	0608	0657	0706	0754	0803	
3	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289	
4	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775	
895	95 1823	1872	1920	1969	2017	2066	2114	2163	2211	2260	
6	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744	48
7	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228	
8	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711	
9	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194	

PROPORTIONAL PARTS

Diff.	1	2	3	4	5	6	7	8	9
51	5.1	10.2	15.3	20.4	25.5	30.6	35.7	40.8	45.9
50	5.0	10.0	15.0	20.0	25.0	30.0	35.0	40.0	45.0
49	4.9	9.8	14.7	19.6	24.5	29.4	34.3	39.2	44.1
48	4.8	9.6	14.4	19.2	24.0	28.8	33.6	38.4	43.2

N.	0	1	2	3	4	5	6	7	8	9	Diff.
900	95 4243	4291	4339	4387	4435	4484	4532	4580	4628	4677	48
1	4725	4773	4821	4869	4918	4966	5014	5062	5110	5158	
2	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640	
3	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120	
4	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601	
905	95 6649	6697	6745	6793	6840	6888	6936	6984	7032	7080	
6	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559	
7	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038	
8	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516	
9	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994	
910	95 9041	9089	9137	9185	9232	9280	9328	9375	9423	9471	
1	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947	
2	9995	*0042	*0090	*0138	*0185	*0233	*0280	*0328	*0376	*0423	
3	96 0471	0518	0566	0613	0661	0709	0756	0804	0851	0899	
4	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374	
915	96 1421	1469	1516	1563	1611	1658	1706	1753	1801	1848	47
6	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322	
7	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795	
8	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268	
9	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741	
920	96 3788	3835	3882	3929	3977	4024	4071	4118	4165	4212	
1	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684	
2	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155	
3	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625	
4	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095	
925	96 6142	6189	6236	6283	6329	6376	6423	6470	6517	6564	
6	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033	
7	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501	
8	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969	
9	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436	
930	96 8483	8530	8576	8623	8670	8716	8763	8810	8856	8903	
1	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369	
2	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835	
3	9882	9928	9975	*0021	*0068	*0114	*0161	*0207	*0254	*0300	
4	97 0347	0393	0440	0486	0533	0579	0626	0672	0719	0765	
935	97 0812	0858	0904	0951	0997	1044	1090	1137	1183	1229	46
6	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693	
7	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157	
8	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619	
9	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082	
940	97 3128	3174	3220	3266	3313	3359	3405	3451	3497	3543	
1	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005	
2	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466	
3	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926	
4	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386	
945	97 5432	5478	5524	5570	5616	5662	5707	5753	5799	5845	
6	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304	
7	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763	
8	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220	
9	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678	

PROPORTIONAL PARTS

Diff.	1	2	3	4	5	6	7	8	9
48	4.8	9.6	14.4	19.2	24.0	28.8	33.6	38.4	43.2
47	4.7	9.4	14.1	18.8	23.5	28.2	32.9	37.6	42.3
46	4.6	9 2	13.8	18.4	23.0	27.6	32.2	36.8	41.4

Table XV—Six-Place Logarithms

N.	0	1	2	3	4	5	6	7	8	9	Diff.
950	97 7724	7769	7815	7861	7906	7952	7998	8043	8089	8135	
1	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591	
2	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047	
3	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503	
4	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958	
955	98 0003	0049	0094	0140	0185	0231	0276	0322	0367	0412	
6	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867	45
7	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320	
8	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773	
9	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226	
960	98 2271	2316	2362	2407	2452	2497	2543	2588	2633	2678	
1	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130	
2	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581	
3	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032	
4	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482	
965	98 4527	4572	4617	4662	4707	4752	4797	4842	4887	4932	
6	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382	
7	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830	
8	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279	
9	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727	
970	98 6772	6817	6861	6906	6951	6996	7040	7085	7130	7175	
1	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622	
2	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068	
3	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514	
4	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960	
975	98 9005	9049	9094	9138	9183	9227	9272	9316	9361	9405	
6	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850	
7	9895	9939	9983	*0028	*0072	*0117	*0161	*0206	*0250	*0294	44
8	99 0339	0383	0428	0472	0516	0561	0605	0650	0694	0738	
9	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182	
980	99 1226	1270	1315	1359	1403	1448	1492	1536	1580	1625	
1	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067	
2	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509	
3	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951	
4	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392	
985	99 3436	3480	3524	3568	3613	3657	3701	3745	3789	3833	
6	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273	
7	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713	
8	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152	
9	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591	
990	99 5635	5679	5723	5767	5811	5854	5898	5942	5986	6030	
1	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468	
2	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906	
3	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343	
4	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779	
995	99 7823	7867	7910	7954	7998	8041	8085	8129	8172	8216	
6	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652	
7	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087	
8	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522	
9	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957	

PROPORTIONAL PARTS

Diff.	1	2	3	4	5	6	7	8	9
45	4.5	9.0	13.5	18.0	22.5	27.0	31.5	36.0	40 5
44	4.4	8.8	13.2	17.6	22.0	26 4	30.8	35.2	39.6
43	4.3	8.6	12.9	17.2	21.5	25.8	30.1	34.4	38 7

ANSWERS

In Chapters I and VIII answers are given to the exercises in the A lists, where two lists occur, and, in general, for all exercises in undivided lists. In Chapters IX and X answers are, with certain exceptions, given for all the exercises. In the Appendix they are given for the odd numbered exercises except in the divided list.

A computing machine was used in making most of the computations. Results obtained by using 6-place logarithms may differ in the 6th significant figure from those given.

CHAPTER I

Pages 5 and 6, list A

1. $I_0 = \$92.62$, $I_e = \$91.36$. **3.** \$50.62. **4.** \$4.04. **5.** 262 days. **6.** 134 days. **7.** 115 days, 299 days. **8.** May 16, 1936 (a leap year). **9.** \$10.02. **10.** \$1133.52. **11.** \$225.10. **12.** Aug. 12, 1933. **14.** \$126.74, \$62.38. **15.** \$850. **16.** \$1048.46.

Pages 11 and 12, list A

1. \$2237.50. **2.** \$758.29. **3.** 7.91%. **4.** June 8. **5.** 30%. **6.** Yes. **7.** 4.17%. **8.** Discount at 6.2%. **9.** 2.04%. **11.** \$3.50. **14.** \$2.81. **15.** \$609.14. **16.** \$390.86. **17.** \$1251.67.

CHAPTER II

Pages 16 and 17, list A

3. (a) \$3992.30, (b) \$251.69, (c) \$6943. **4.** \$5101.11. **5.** \$507.34. **6.** \$34,939. **7.** \$1707.50. **8.** 202,460. **9.** \$3536.15. **10.** \$2346.78.

Pages 22 to 24, list A

1. (d) \$276.36, (e) \$7466.41, (f) \$39,786. **2.** \$73.91, \$86.27. **3.** (a) \$3022.20, (b) \$3021.23. **4.** \$349.73. **5.** \$1336.07. **7.** 8 years, 111 days. **8.** 14 years, 76 days; 14 years, 74 days, respectively, at 5%. 10 years, 89 days, 10 years, 87 days, respectively, at 7%. 365 days used for a year. **9.** 10 years, 246 days; 10 years, 247 days, respectively. **10.** March 20, 1930. **11.** 6.68% **12.** 4.26%. **13.** 2.92%, \$91.73, \$86.60. **15.** 23.45 years, 14.21 years, 10.25 years, respectively (using formula).

Pages 30 and 31, list A

1. \$1516.89 **2.** Yes. **3.** \$1123.04. **4.** 12 years, 328 days. **5.** 4.14%, \$101.98. **6.** 2.01% **7.** 4.89%, 2.23%. **8.** 6.17%, 3.04% **9.** 17 years, 57 days. **10.** 7 years, 16 days. **11.** 3.65% nominal, 3.71% effective. **12.** Yes, yes, no. **13.** Yes, yes, no. **16.** \$635.50.

Pages 36 and 37

1. (a) \$2771.54, (b) \$1175.29, (c) \$328.81. **3.** (a) \$236.69, (b) \$1360.23, (c) \$800.84, (d) \$6741.60. **4.** \$245.20, \$29.80. **5.** \$1971.23, \$1979.33. **6.** \$7045.96. **7.** Yes. **8.** C paid \$289.95, B lost \$3.76. **9.** \$1846. **10.** \$374.46.

CHAPTER III

Page 41

1. $5923.91. **2.** $601.48 (compound interest formula). **3.** $4889.97. **4.** $2442.52. **5.** $2532.82. **6.** Yes. **7.** .306 year with exact formula, compound interest; .307 year with approximate formula. **8.** 5.41 years, 5.50 years.

CHAPTER IV

Pages 47 and 48, list A

1. $482.80. **2.** $2865.97. **3.** $36,446.05. **4.** $1225.04. **5.** $3579.33. **6.** $42,919.74. **7.** $6314.09. **8.** $1014.38. **9.** $6123.35. **10.** 110.9 **11.** 43.67.

Pages 53 and 54, list A

3. $11,613.54. **4.** $5086.28. **6.** $2017.82, $1259.57. **7.** $5707.84. **10.** 434.093.

Page 57

1. $772.44. **2.** $1462.81. **3.** $11,254.17. **4.** $5735.68. **5.** $5068.32. **6.** $9859.82. **7.**$79,943.42.

Pages 59 and 60, list A

1. 17 full payments, and an 18th payment of $76.59. **2.** 21 months. **3.** 15 years, $4882. **4.** 280 days after 11th deposit, using exact simple interest for the fraction of a year. **5.** 4.180%. **6.** 4.676%. **7.** $271.75. **9.** 19 withdrawals of $250, and a 20th withdrawal of $119.46. **10.** 12 payments (equivalent to a cash price of $10,207). **11.** 4.833%. **12.** Rates are 5.33%, and 4.47%, respectively.

CHAPTER V

Page 66

1. $23,287.02. **2.** $22,976.10. **3.** $7924.19. **4.** $6064.52. **9.** Account after 18th deposit, $11,731.00. **10.** Five withdrawals of $500 each exhaust $2132.06 of original deposit. The balance of $367.94 produces a 6th withdrawal of $458.24.

Pages 70 to 73

1. $6465.66 **2.** $1258.85. **3.** $8422.89. **4.** $23,799.13. **5.** $4514.61, $4426.08, $4448.07 **6.** $4076.95. **7.** $96.37. **11.** $16,064.28. **14.** At the end of 12 years (11½ years after first deposit), when a 24th deposit of $531.80 must be made. **15.** 113 withdrawals of $250, and a 114th withdrawal of $21.76. **16.** $55.09. **17.** $296.57, $343.26. **18.** $12,177.02. **19.** $30,282.07. **20.** $7250.32. **21.** $3838.38. **22.** $16,362.41. **23.** $6634.85. **25.** 2.63%. **27.** 3.80%. **29.** $18,393.79.

CHAPTER VI

Pages 78 and 79, list A

2. $2237.77, $3017.77. **3.** $1437.40, $28,194.24. **4.** $2889.17. **5.** $84.35. **6.** $1654.48. **7.** $17,754.82. **8.** $127,726.65. **9.** $331.71. **10.** $6650.62.

Pages 83 to 85, list A

1. $1449.32. **2.** $333.97. **3.** $519.51. **4.** $116.12. **5.** $7174.13. **6.** $721.04. **7.** $5765.19 book value, and $14,234.81 returned to capital. **9.** $2981.82, $3116.66. **10.** $1715.25. **11.** $2492.37. **12.** $740.40. **13.** $436.32. **14.** $3996.94. **15.** $39,152.92. **16.** $52,925.10.

CHAPTER VII

Page 98

2A. (1) $1034.76, (2) $479.79, (3) $118.52, (4) $1043.85, (5) $533.94, (6) $116.74.

Page 101

1. $540.78. **2.** $1086.46 (exact interest) **3.** $90.40 (exact interest) **4.** $568.60 (exact interest).

Page 102

4. $98.43. **5.** $544.51.

Pages 105 and 106, list A

1. Book value, $10,455.07; toward amortization, $274.68. **3.** $48,281.52. **4.** $987.05, after interest had been received. **5.** $10,429.25. **6.** $48,414.97.

Page 108, list A

1. (1) 6.182%, (2) 3.534%, (3) 6.059%, (4) 4.297%, (5) 4.266%. **2.** 4.46%. **3.** 4.39.%.

CHAPTER VIII

Pages 115 and 116, list A

1. Book value, $5433.33; towards depreciation, $3593.33. **3.** Book values· $8148.12, $5532.65, $3756.72, $2550.85; percentage annually written off, 32.10% **4.** $28,457.97. **5.** $17,168.79. **6.** $11,487.

Pages 119 and 120, list A

1. $1.92. **2.** $1303. **3.** $2844. **4.** $20,156. **5.** 14,875.

Pages 121 and 122, list A

1. 22 years. **2.** 19.2 years, 19.0 years, 19.5 years. **4.** 19.7 years.

Page 124, list A

1. $671,407. **2.** $5.31. **3.** $18,598.91. **4.** Every 5 years, $1876.86.

Page 126

(a) $127,969, (b) $95,410.

CHAPTER IX

Page 129

1. 9383. **2.** 89.

Page 133

2. $829.68, $908.96. **4.** $2737.93.

Pages 134 and 135

2. $16,082.94, $17,849.61. **4.** $461.36. **8.** $68.42.

Page 136 and 137

1. 194 receive no payment, 12 receive just one payment. **2.** $3349.23. **3.** Life annuity $2239.19 less. **6.** $10,103.38. **11.** Refund annuity of $226.09 (using net cost) with 18 guaranteed payments; ordinary life annuity of $248.71.

Pages 139 to 141

1. $487.97. **2.** $328.04.

Pages 142 and 143

2. $245.34. **3.** $2110.22. **4.** $18.33. **5.** $21.16.

Pages 144 and 145

American Experience Table of Mortality used. **1.** (a) .7150, (b) .0192. **3.** 220. **6.** (a) .0097.

CHAPTER X

Pages 150 and 151

5. $193.15. **6.** $101.88. **10.** $157.44.

Pages 152 and 153

1. $25.91. **2.** $7.83, for age 25; $9.51 for age 40.

Page 154

1. $206.02. **3.** $186.24. **6.** $228.07.

Pages 159 to 161

2. $183.27. **3.** $477.50.

Page 162

1. $3424.60. **3.** 17.7 years.

APPENDIX

Pages 179 and 180

1. 15. **3.** 680. **5.** $x^6 - 12x^5y + 60x^4y^2 - 160x^3y^3 + 240x^2y^4 - 192xy^5 + 64y^6$. **7.** $a^7 + 7a^6x + 21a^5x^2 + 35a^4x^3 + 35a^3x^4 + 21a^2x^5 + 7ax^6 + x^7$. **9.** 96,059,601. **11.** 992,023,968,016. **13.** 1.2299. **15.** 1.1314. **17.** $1 + \frac{1}{5}x - \frac{2}{25}x^2 + \frac{6}{125}x^3 - \frac{21}{625}x^4 + \ldots$. **19.** 6.325. **21.** 3.107. **27.** 1.0309278343.

Page 182

3. 14,175. **5.** $a = 3, d = 5$. **7.** 23. **9.** $s = \frac{(l - a + d)(l + a)}{2a}$.

Page 184

1. 765. **3.** $\frac{189}{16}$. **5.** $-\frac{3}{2}$. **7.** $-\frac{17}{6}$.

Page 189

1. 1.86 6069. **3.** 8.63 5645 − 10. **5.** 5.81 5833. **7.** 2.40 3573.
9. 0.01 0439.

Page 190

1. 21.1590. **3.** 2.8205. **5.** 0.13 2597.

Page 193, list A*

1. 15,581.0. **2.** 3252 25. **3.** 18 6822. **4.** 2.50026. **5.** 228 905.
6. .200078. **7.** 11,344. **8.** 2178 8. **9.** 247.918. **10.** 3139.68.
11. 35,985. **12.** 2308 59. **13.** 048895. **14.** 1 01654.

* Answers are given only to 5 figures where a 6-place logarithm computation is uncertain beyond that point. The 5th figure in Exercise 11 is uncertain. The answers for Exercises 12, 13, 14 were taken from interest tables.

INDEX

Accumulated amount, of an annuity, 45
Accumulated cost, of insurance, 152
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